Matching in Nonrelativistic Effective Quantum Field Theories

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They began to weave curtains of darkness.
They erected large pillars round the Void,
With golden hooks fastend in the pillars;
With infinite labour the Eternals
A woof wove, and called it Science.

William Blake, The Book of Urizen

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Chapter 1

Introduction

Relativistic quantum field theories (RQFTs) describe the interaction of particles at energies accessible in today's experiments. In most cases, exact solutions are not known and one has to resort to perturbation theory or lattice calculations. The former is only valid at energies where the interaction is small (in the asymptotic region). In an infrared free theory like QED, there can be an additional problem when the energies and momenta of the process under consideration become small. Consider, for example, the differential cross section of e^+e^- scattering in the center of mass system to leading order. Expanding in the relative velocity v one finds that it diverges like $1/v^4$. The reason for this nonsensical result is the fact that perturbation theory breaks down at a scale of the order of $m_e\alpha^2$ – one would have to sum infinitely many graphs that all give contributions of the same order of magnitude. Non-perturbative effects of this kind are notoriously difficult to handle in a RQFT.

In this particular process, the particles can form a bound state which shows up as an isolated pole of the fermionic four-point function in the center of mass momentum, which cannot be seen to any finite order in ordinary perturbation theory. The tool to study this object is the so-called homogeneous Bethe-Salpeter equation. It is a fourth-order integro-differential equation for the "wave function", which is essentially the residue of the pole. This is a rather complicated object and no methods are known to solve this difficult mathematical problem exactly.

All approaches to solve the Bethe-Salpeter equation perturbatively take advantage of the fact that the scale $m_e\alpha^2$ is much smaller than m_e, suggesting that a non-relativistic approximation is a good starting point. It turns out that this procedure suffers from numerous technical problems and despite the long history of the topic, there is to date no truly systematic perturbation theory available.

Caswell and Lepage [1] pointed out that the traditional approach is not well adapted to the non-relativistic nature of the problem. After all, simple quantum mechanics gives the energy levels of positronium quite accurately. They recognized that the source of all the problems is the existence of a hierarchy of physical scales: the electron mass m_e, the typical bound state momentum $m_e\alpha$ and the bound state

energy $m_e\alpha^2$. In the relativistic treatment, all of these scales are present in the integral kernel of the Bethe-Salpeter equation and it is very difficult to expand it systematically. They suggest that one should first construct an effective theory, in which the physics that takes place at scales of the order of m_e or higher are represented by local interactions of the fields, which are suppressed by powers of $1/m_e$. The coefficients of these terms are determined by comparing scattering amplitudes with those of the full theory at energies where bound states can be neglected. With the information about high energies encoded in the effective couplings, one can then perform bound state calculations with the effective theory.

The point is that the remaining physical scales are much smaller than m_e. As a consequence, no additional heavy particles (of mass m_e) can be created[1] and the theory is confined to a subspace of the Fock space in which their number is conserved. This is precisely the setting one has in quantum mechanics: As long as we don't try to resolve processes taking place at a scale of the order of the Compton wavelength $1/m_e$, the description in terms of a wave function that obeys a Schrödinger equation is perfectly adequate.

The concept of a non-relativistic quantum field theory (NRQFT) outlined above is the bridge between field theory and quantum mechanics. It is equivalent to the full theory below the heavy scale but takes advantage of the non-relativistic character of some degrees of freedom by incorporating relativistic effects in a systematic expansion in inverse powers of some heavy scale M. The interaction of heavy particles is described by a Schrödinger equation whose Hamilton operator is obtained from the Lagrangian of the NRQFT. What seemed so hard to do in the RQFT, namely the summation of the non-perturbative part of the theory, simply amounts to solving a lowest order approximation of this Schrödinger equation exactly. One can then use standard methods of quantum mechanics to perform a systematic perturbation theory from there.

The formalism has been applied to various processes with considerable success (the following references are only a selection and by no means complete). NRQCD, the low-energy version of QCD, was used to study bound states of heavy quarks by Bodwin, Braaten and Lepage [2]. Muonium and Positronium hyperfine splitting was already considered in [1] and later extended to higher order corrections [3, 4, 5, 6, 7, 8].

Another system where a NRQFT approach can be useful is the bound state formed by π^+ and π^-. Because the binding energy is of the order of keV, it probes the $\pi\pi$ interaction practically at threshold. The decay width of this atom is related to the $\pi\pi$ scattering lengths and will be measured soon in the DIRAC experiment at CERN [9], providing a high precision test of low energy QCD. The leading term of the lifetime was given by Uretsky and Palfrey [10]. Recently, corrections have been calculated using different techniques to solve the Bethe-Salpeter equation in

[1] In positronium, e^+ and e^- can, however, annihilate into photons. We don't want to go into this rather subtle issue here and ignore this effect. To be save, we could consider a stable system, like $e^+\mu^-$ as is actual done in [1], introducing another scale m_μ.

the relativistic framework [11, 12, 13, 14, 15]. Another approach, based on non-relativistic potential models, was pursued by the authors of refs. [16, 17, 18]. First attempts using a NRQFT approach have been published [19, 20] but need further clarification.

Let us also mention that there is a different branch of NRQFTs, where there is only one heavy particle involved. In this case, the scales $m_e\alpha$, $m_e\alpha^2$ are absent and with them the non-perturbative effects. The heavy particle can be considered to be static and power counting becomes very simple. This version of a NRQFT is used for the description of mesons containing one heavy quark under the name of heavy quark effective theory (HQET) and also for the pion-nucleon system where it is called heavy baryon chiral perturbation theory (HBCHPT). See refs. [21, 22] for reviews on these subjects.

A crucial step in the construction of a NRQFT is the matching with the fundamental theory, where the coupling constants are adjusted such that the scattering amplitudes agree to some order in inverse powers of the heavy scale. In order to do this, one needs to renormalize both theories, i.e. introduce a regularization scheme that allows to absorb the divergences of Green's functions into the coupling constants in a systematic way. Also, one has to express the physical mass of the heavy particle in terms of the parameters of the theory and determine the effective normalization of the field (the "wave function renormalization") due to self energy effects. This is certainly no problem in the RQFT, which is expressed in a Lorentz covariant form. The NRQFT is not covariant and it is not a priori clear how these tasks should be performed there. The fact that it took some time for people to realize that in some versions of HQET and HBCHPT the fields were incorrectly normalized even at tree level [23, 24] shows that this question is not as innocent as it may seem. Unfortunately, the discussions are often obscured by the formalism of the particular model under consideration.

However, as in a RQFT, the procedure of mass and wave function renormalization is independent of a particular model and can be treated once and for all in the language of the one-particle irreducible two-point function. To the best of the author's knowledge, such a discussion is not available in the literature. The present work tries to fill this gap by studying how this mechanism works in the case of a heavy scalar field. We only consider Yukawa-type couplings to other scalar fields to avoid complications due to gauge symmetry and spin.

This work is organized as follows. In chapter 2 we show how amplitudes and Green's functions of a generic Lagrangian with one heavy scalar field can be matched with the corresponding effective theory. In chapter 3, we consider a toy-model and explicitly construct two non-local effective Lagrangians that are equivalent to the full theory in the pure particle- and anti-particle sectors, verifying the general statements made in chapter 2. Finally, the $1/M$ expansion of tree-level Green's functions and Amplitudes in the full theory is discussed in chapter 4 and it is shown how they can be reproduced by the effective theory order by order in powers of the inverse heavy scale.

Chapter 2

Matching in the Particle Sector

2.1 Transition Amplitudes

To have a specific example and to keep things simple at the same time, we consider a theory of the form

$$\begin{aligned} \mathcal{L} &= \mathcal{L}^0 + \bar{\mathcal{L}}^0 + \mathcal{L}^{\text{int}} \\ \mathcal{L}^0 &= \partial_\mu H^* \partial^\mu H - M^2 H^* H. \end{aligned} \tag{2.1}$$

Here, $\bar{\mathcal{L}}^0$ contains the kinetic part of all the fields that interact with H through the interaction Lagrangian $\mathcal{L}^{\text{int}}$. We assume that the masses of these fields are all much smaller than M, i.e. H is the only heavy degree of freedom. As such, they appear unaltered in the effective theory that describes physics at a scale much smaller than M. Therefore, we first concentrate on processes among heavy particles alone.

The free Lagrangian of the heavy field has a $U(1)$ symmetry and the particles carry a charge which is conserved in all processes if $\mathcal{L}^{\text{int}}$ respects this symmetry. We shall refer to the two types of field quanta as particle- and anti-particle. They enter the free Lagrangian symmetrically and can only be distinguished by the interaction with an external field. Scattering processes which are related by crossing are described by the same invariant amplitude.

2.1.1 Relativistic Theory

The fundamental objects we have to study are the connected Green's functions

$$G^{(2n)}(x, y) = \langle 0 | T \hat{H}(x) \hat{H}^\dagger(y) | 0 \rangle_c. \tag{2.2}$$

Here, x, y are vectors $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$ and we use the notation $\hat{f}(x) \equiv f(x_1) \ldots f(x_n)$. Further notation is given in appendix A. To each external momentum corresponds a two-point function $G^{(2)}$ and we define the truncated function $G_{tr}^{(2n)}$ by

$$G^{(2n)}(p, q) = \hat{G}^{(2)}(p) \hat{G}^{(2)}(q) G_{tr}^{(2n)}(p, q). \tag{2.3}$$

Each of the factors $G^{(2)}(p_i)$ has a pole when the momentum is on the mass shell $p_i^2 = M_{\mathrm{p}}^2$, where M_{p} is the physical mass of the particle. The scattering amplitude, involving $2n$ heavy particles in this case, is related to the residue of the multiple pole when all momenta are put on their mass shells. The precise relation is given by the LSZ formalism summarized in appendix E for the case at hand. Applied to the process with n heavy particles in the initial and final states

$$\begin{aligned}\langle p_1, \dots, p_n; \text{out}|q_1, \dots, q_n; \text{in}\rangle = \langle p_1, \dots, p_n; \text{in}|q_1, \dots, q_n; \text{in}\rangle \\ + i(2\pi)^4 \delta^4 (P - Q)\, T_{n\to n}, \quad (2.4)\end{aligned}$$

where $P = \sum_{i=1}^n p_i$ and $Q = \sum_{i=1}^n q_i$, we find

$$T_{n\to n} = \frac{1}{i} Z_H^n \, G_{tr}^{(2n)}(p,q)\Big|_{\text{on-shell}} . \quad (2.5)$$

"On-shell" means $p_i^0 = \omega_{\mathrm{p}}(\mathbf{p}_i) = \sqrt{M_{\mathrm{p}}^2 + \mathbf{p}_i^2}$, $q_i^0 = \omega_{\mathrm{p}}(\mathbf{q}_i)$ and Z_H is the residue of the two-point function $G^{(2)}$. Note that, due to the manifest covariance of the theory, this quantity transforms as a scalar under the Lorentz group.

In such a process, heavy anti-particles are only involved as virtual states. Therefore, it should be possible to remove them as an explicit degree of freedom and incorporate them into the interaction.

2.1.2 Separating Particles and Anti-Particles

The first step towards this goal is to separate particles and anti-particles in the free field. Consider the equation of motion

$$(\Box + M^2)H = 0 \quad (2.6)$$

obtained from $\mathcal{L}^0$. The most general solution is a superposition of plane waves

$$H(x) = \int \frac{d^3p}{(2\pi)^3 2\omega(\mathbf{p})} \left(a(p)e^{-ipx} + b^*(p)e^{ipx} \right). \quad (2.7)$$

To separate the positive and negative frequency contributions, we define the differential operators (see also appendix B)

$$D_\pm = \pm i\partial_t - \sqrt{M^2 - \Delta} \quad (2.8)$$

$$d = (2\sqrt{M^2 - \Delta})^{-\frac{1}{2}} \quad (2.9)$$

and set

$$H_\pm = -D_\mp dH. \quad (2.10)$$

With this choice we have

$$H_+(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega(\mathbf{p})}} a(p)e^{-ipx} \tag{2.11}$$

$$H_-(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega(\mathbf{p})}} b^*(p)e^{ipx} \tag{2.12}$$

and

$$H = d(H_+ + H_-). \tag{2.13}$$

The operator d is not really necessary for this decomposition and was only introduced for later convenience. The fields $H_\pm$ satisfy the equations

$$D_\pm H_\pm(x) = 0, \tag{2.14}$$

which are the Euler-Lagrange equations of the Lagrangians

$$\mathcal{L}^0_\pm = H^*_\pm D_\pm H_\pm. \tag{2.15}$$

After canonical quantization, the operators $H^\dagger_+$ and H_- create a particle and an anti-particle state, respectively (see appendix C).

2.1.3 Effective Theory in the Particle Sector

A theory for the particle sector should therefore be of the form

$$\mathcal{L}_+ = \mathcal{L}^0_+ + \bar{\mathcal{L}}^0 + \mathcal{L}^{\text{int}}_+. \tag{2.16}$$

In appendix E it is shown that the Fock space of the free heavy particles, in which the incoming and outgoing particles live, is the same as in the relativistic theory. This is obviously a necessary condition for the existence of an interpolating field that should reproduce transition amplitudes of a relativistic theory.

The interaction Lagrangian is a local function of the fields and their derivatives and can be written as

$$\mathcal{L}^{\text{int}}_+ = \sum_{\nu=1}^{\infty} \frac{1}{M^\nu}\mathcal{L}^\nu_+, \tag{2.17}$$

where $\mathcal{L}^\nu_+$ contains ν space or time derivatives. This means that we deal here with an effective field theory in which M is considered to be a hard scale. It can only describe processes in which all relevant scales are much smaller than that. In practice, one always truncates the Lagrangian at some power in $1/M$ but for the sake of the following arguments, let us assume that we have summed up the contributions to all orders and postpone the discussion of this issue.

$U(1)$ symmetry of the Lagrangian insures that the heavy field only occurs in the combination $H^\dagger_+ H_+$, which means that the number of heavy particles is conserved

at each vertex (H_+ destroys an incoming particle and $H_+^\dagger$ creates an outgoing one) and therefore for any process (this is simply the consequence of charge conservation when there is only one type of charge). The theory is thus naturally confined to a subspace of the Fock space in which the number of heavy particles is fixed.

We can start by writing down the most general interaction Lagrangian which respects the symmetries of $\mathcal{L}$. However, we can immediately see that Lorentz symmetry is already violated by $\mathcal{L}_+^0$. The question is then, how much of this symmetry we have to incorporate into $\mathcal{L}_+$ to be able to calculate a transition amplitude with the correct transformation properties under the Lorentz group. Let us formulate a pragmatic approach to the problem.

Due to the lack of knowledge of the transformation properties of H_+ under the Lorentz group[1], we only require rotational invariance of the Lagrangian. We can then calculate the connected Green's functions

$$G_+^{(2n)}(x,y) = \langle 0|T\hat{H}_+(x)\hat{H}_+^\dagger(y)|0\rangle_c. \tag{2.18}$$

Next, we can try to derive a reduction formula for this theory, relating transition amplitudes to poles of these Green's functions. As shown in appendix E, this involves one non-trivial assumption about the structure of the two-point function, namely that it permits the definition of a physical mass M_p so that

$$G_+^{(2)}(p) = \frac{1}{i}\frac{Z_+(\mathbf{p}^2)}{\omega_\mathrm{p}(\mathbf{p}) - p^0 - i\epsilon} + \ldots \tag{2.19}$$

This implies that not all of the coupling constants of the original Lagrangian are independent. No additional assumptions are needed to define the object

$$T^+_{n\to n} = \frac{1}{i}\prod_{i=1}^{n} Z_+(\mathbf{p}_i^2)^{\frac{1}{2}} Z_+(\mathbf{q}_i^2)^{\frac{1}{2}}\; G^{(2n)}_{+,tr}(p,q)\Big|_{\text{on-shell}}, \tag{2.20}$$

where the truncated function is defined by

$$G_+^{(2n)}(p,q) = \hat{G}_+^{(2)}(p)\hat{G}_+^{(2)}(q)G_{tr}^{(2n)}(p,q). \tag{2.21}$$

$T^+_{n\to n}$ does not yet transform as a scalar under the Lorentz group as it should if it is supposed to reproduce $T_{n\to n}$.

2.1.4 Matching

Symmetry only fixes each term in the Lagrangian up to a factor. These low energy constants (LEC) are at our disposal and can be chosen in such a way that all scattering amplitudes considered above are identical. This procedure is called matching. Before we formulate it, we should say a word about the normalization

[1] In HQET and HBCHPT one introduces a four-velocity v_μ to write down the Lagrangian in different frames of reference and Lorentz invariance is replaced by "reparametrisation invariance" [25]

of one-particle states, because the amplitudes clearly depend on them. Although arbitrary, there is still a most natural choice of normalization (see also appendix C). In the full theory, we chose it to be Lorentz invariant

$$\langle p|p'\rangle = \langle \bar{p}|\bar{p}'\rangle = 2\omega_{\mathrm{p}}(\mathbf{p})(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}'), \tag{2.22}$$

where as for $\mathcal{L}_+$ we chose

$$\langle p|p'\rangle = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}'). \tag{2.23}$$

Therefore, before we try to match amplitudes, we must make up for this difference in normalization by replacing, say, the states used in the effective theory by

$$|p\rangle \to \sqrt{2\omega_{\mathrm{p}}(\mathbf{p})}|p\rangle. \tag{2.24}$$

The matching condition then reads

$$T_{n\to n} = \prod_{i=1}^{n} \sqrt{2\omega_{\mathrm{p}}(\mathbf{p}_i)}\sqrt{2\omega_{\mathrm{p}}(\mathbf{q}_i)}T^{+}_{n\to n}, \tag{2.25}$$

which automatically restores Lorentz symmetry for the transition amplitudes. In the last section we have seen that the effective theory is actually an expansion in inverse powers of the heavy scale M. The matching can only make sense if the relativistic amplitude possesses such an expansion in the region of phase space we are interested in.

2.2 Green's Functions

The matching of scattering amplitudes involves only Green's functions evaluated on the mass shell of all particles involved. They are, however, also interesting in the unphysical region because they reflect general properties of quantum field theories like unitarity in their non-trivial analytic structure. It is interesting to see how the Green's functions of the fundamental theory compare to the ones of the effective theory.

Being unphysical quantities, off-shell Green's functions have no unique definition. Redefinitions of the fields that do not change the classical field theory give, in general, different off-shell results while describing the same physics. Suppose we have chosen a particular off-shell extrapolation in the fundamental theory. Naively, one may be tempted to identify the truncated functions $G^{(2n)}_{tr}$ with $G^{(2n)}_{+,tr}$, i.e. consider the latter to be the $1/M$ expansion of the former. One would then expect that they differ only by a polynomial in the momenta which can be absorbed by a proper choice of coupling constants in the effective theory. However, this is not true, as we will show now.

Two remarks about the following statements are in order. First, we suppose that renormalization was performed in both theories and that everything is finite

and well defined. Second, as mentioned before, the effective theory is an expansion in $1/M$. Therefore, the matching is actually performed order by order in $1/M$ and we assume that the relativistic Green's functions can be expanded in this way.

Let us start with eq. (2.25). It can be written in terms of the truncated Green's functions as

$$Z_H^n \; G_{tr}^{(2n)}(p,q)\Big|_{\text{on-shell}} = \prod_{i=1}^{n}(Z_+(\mathbf{p}_i^2)\omega_{\rm p}(\mathbf{p}_i))^{\frac{1}{2}}(Z_+(\mathbf{q}_i^2)\omega_{\rm p}(\mathbf{p}_i))^{\frac{1}{2}} \; G_{+,tr}^{(2n)}(p,q)\Big|_{\text{on-shell}}. \quad (2.26)$$

Without knowing the relationship between the residues Z_H and Z_+, we cannot express, say, $G_{tr}^{(2n)}$ in terms of quantities that can be calculated with $\mathcal{L}_+$ alone. In appendix D it is shown how such a relation emerges from the matching of the two-point functions. The statement is that when the irreducible parts Σ, Σ_+ defined by

$$G^{(2)}(p) = \frac{1}{i}\frac{1}{M^2 - p^2 + i\Sigma(p^2) - i\epsilon} \quad (2.27)$$

$$G_+^{(2)}(p) = \frac{1}{i}\frac{1}{\omega(\mathbf{p}) - p^0 + i\Sigma_+(p^0,\mathbf{p}^2) - i\epsilon}, \quad (2.28)$$

are matched according to

$$\Sigma_+(p^0,\mathbf{p}^2) = \frac{\Sigma(p^2)}{2\omega(\mathbf{p}) + \frac{i\Sigma(p^2)}{\omega(\mathbf{p})+p^0}}, \quad (2.29)$$

the physical masses defined by

$$M_{\rm p} = M^2 + i\Sigma(M_{\rm p}^2) \quad (2.30)$$

$$\omega_{\rm p}(\mathbf{p}) = \sqrt{M_{\rm p}^2 + \mathbf{p}^2} = \omega(\mathbf{p}) + i\Sigma_+(\omega_{\rm p}(\mathbf{p}),\mathbf{p}^2) \quad (2.31)$$

are identical and the residues of

$$G^{(2)}(p) = \frac{1}{i}\frac{Z_H}{M_{\rm p}^2 - p^2 - i\epsilon} + \text{regular}, p^2 \to M_{\rm p}^2 \quad (2.32)$$

$$G_+^{(2)}(p) = \frac{1}{i}\frac{Z_+(\mathbf{p}^2)}{\omega_{\rm p}(\mathbf{p}) - p^0 - i\epsilon} + \text{regular}, p^0 \to \omega_{\rm p}(\mathbf{p}) \quad (2.33)$$

are related by

$$Z_+(\mathbf{p}^2) = \frac{(\omega(\mathbf{p}) + \omega_{\rm p}(\mathbf{p}))^2}{4\omega_{\rm p}(\mathbf{p})\omega(\mathbf{p})} Z_H. \quad (2.34)$$

If we plug this into eq. (2.26), we find

$$G_{tr}^{(2n)}(p,q)\Big|_{\text{on-shell}} = \prod_{i=1}^{n}\frac{\omega(\mathbf{p}_i) + \omega_{\rm p}(\mathbf{p}_i)}{\sqrt{2\omega(\mathbf{p}_i)}}\frac{\omega(\mathbf{q}_i) + \omega_{\rm p}(\mathbf{q}_i)}{\sqrt{2\omega(\mathbf{q}_i)}} \; G_{+,tr}^{(2n)}(p,q)\Big|_{\text{on-shell}} \quad (2.35)$$

and all quantities on the r.h.s. can be calculated with the Lagrangian $\mathcal{L}_+$. Let us extend this relation to off-shell Green's functions. For this purpose we define a new truncation procedure

$$G_+^{(2n)}(p,q) = \hat{\mathcal{G}}_+(p)\hat{\mathcal{G}}_+(q)\bar{G}_{+,tr}^{(2n)}(p,q) \tag{2.36}$$

with

$$\mathcal{G}_+(p) \doteq \frac{G_+^{(2)}(p)}{\sqrt{2\omega}}\left(1 - \frac{i\Sigma_+(p^0,\mathbf{p}^2)}{\omega(\mathbf{p})+p^0}\right) \tag{2.37}$$

and impose the off-shell matching condition

$$G_{tr}^{(2n)}(p,q) = \bar{G}_{+,tr}^{(2n)}(p,q), \tag{2.38}$$

which indeed reduces to eq (2.35) on the mass shell. The functions $G_{+,tr}^{(2n)}$ and $\bar{G}_{+,tr}^{(2n)}$ differ essentially by the self-energy Σ_+, which is a non-trivial function of momentum. This is the reason why a matching between the “naturally” truncated functions $G_{tr}^{(2n)}$ and $G_{+,tr}^{(2n)}$ is impossible - they differ by more than just a polynomial. This point will be illustrated in section 3.3 in a simple toy-model.

Chapter 3

Construction of the Effective Lagrangian for a simple Model

3.1 The Model

The model we are considering is given by

$$\begin{aligned} \bar{\mathcal{L}}^0 \equiv \mathcal{L}_l^0 &= \frac{1}{2}\partial_\mu l \partial^\mu l - \frac{m^2}{2} l^2 \\ \mathcal{L}^{\text{int}} &= eH^* H l \end{aligned} \tag{3.1}$$

in the notation of section 2.1. To stay in the scope of that section we chose $m \ll M$ and refer to l as the light field. It will always keep its relativistic form. In the following, we will explicitly construct an effective theory of the form given in eq. (2.16) that can be proven to reproduce the scattering amplitudes in the sector where there is a fixed number of heavy particles and an arbitrary number of light particles.

3.2 Interaction with an External Field

In this section, the light field l is a given function of space and time and we consider the Lagrangian

$$\mathcal{L}^{ext} = \mathcal{L}^0 + \mathcal{L}^{\text{int}} + j^* H + H^* j. \tag{3.2}$$

The equation of motion

$$D_e H \doteq (D_M - el)H = j, \tag{3.3}$$

where $D_M = \Box + M^2$, has the formal solution

$$H = D_e^{-1} j. \tag{3.4}$$

D_e^{-1} is the complete two-point function of this theory and can be expressed in terms of the free propagator D_M^{-1} defined in appendix B as

$$D_e^{-1} = D_M^{-1} \frac{1}{1 - elD_M^{-1}}. \tag{3.5}$$

We define the truncated two-point function T by

$$D_e^{-1} = D_M^{-1} + D_M^{-1} T D_M^{-1}. \tag{3.6}$$

In perturbation theory, it is simply a string of free propagators with insertions of the external field

$$T = el + e^2 l D_M^{-1} l + O(e^3). \tag{3.7}$$

All information about a particle moving in the external field is contained in this operator. The possible physical processes are the scattering of a particle or an anti-particle (including the formation of bound states if the external field allows them), pair-annihilation and pair-creation. We are about to construct two independent non-local theories that can reproduce the scattering processes for particles and anti-particles separately. To this end, we define the fields $H_\pm$ as in eq. (2.10) and introduce the vectors

$$\vec{H} = \begin{pmatrix} H_+ \\ H_- \end{pmatrix} \qquad \vec{j} = \begin{pmatrix} j \\ j \end{pmatrix} \tag{3.8}$$

and the operator

$$\begin{aligned} D &= \begin{pmatrix} A & eB \\ eB & C \end{pmatrix} \\ A &= D_+ + eB \\ C &= D_- + eB \\ B &= dld. \end{aligned} \tag{3.9}$$

It is easy to check that $\vec{H}$ obeys

$$D\vec{H} = -d\vec{j}. \tag{3.10}$$

Writing D^{-1} as

$$D^{-1} = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} \tag{3.11}$$

and using the fact that j is arbitrary, we find the operator identity

$$D_e^{-1} = -\sum_{n=1}^{4} dG_n d. \tag{3.12}$$

To explore the significance of this, we investigate the structure of the G_n. They can be expressed in terms of the A, B, C defined above by solving the equation $DD^{-1} = \mathbf{1}$. Their structure in terms of the Green's functions $D_\pm^{-1}$ allows for a definition of truncated objects $T_{\pm\pm}$ just like in (3.6)

$$G_1 = (A - e^2 BC^{-1}B)^{-1} \doteq D_+^{-1} - D_+^{-1}T_{++}D_+^{-1} \tag{3.13}$$

$$G_2 = -eA^{-1}BG_4 \doteq -D_+^{-1}T_{+-}D_-^{-1} \tag{3.14}$$

$$G_3 = -eC^{-1}BG_1 \doteq -D_-^{-1}T_{-+}D_+^{-1} \tag{3.15}$$

$$G_4 = (C - e^2 BA^{-1}B)^{-1} \doteq D_-^{-1} - D_-^{-1}T_{--}D_-^{-1}. \tag{3.16}$$

It is straight forward to show that the $T_{\pm\pm}$ are all essentially equal to T (see appendix F). More precisely, we find that

$$T_{++} = T_{+-} = T_{-+} = T_{--} = dTd \tag{3.17}$$

holds to all orders in perturbation theory. We have therefore found a decomposition of the r.h.s. of eq. (3.6) in which each of the four pieces contains the complete truncated function T. We define a non-local Lagrangian for each of the fields $H_\pm$ by

$$\begin{aligned} \mathcal{L}_\pm^{ext} &= H_\pm^* \mathcal{D}_\pm H_\pm \\ \mathcal{D}_+ &= A - e^2 BC^{-1}B \\ \mathcal{D}_- &= C - e^2 BA^{-1}B. \end{aligned} \tag{3.18}$$

The associated two-point functions

$$\langle 0|TH_+(x)H_+^\dagger(y)|0\rangle = iG_1(x,y) \tag{3.19}$$

$$\langle 0|TH_-(x)H_-^\dagger(y)|0\rangle = iG_4(x,y) \tag{3.20}$$

contain all the information about the interaction of one particle and one anti-particle with the external field, respectively. Note that pair creation or annihilation processes are not included: the fields $H_\pm$ do not talk to each other.

Let us illustrate the connection between the original Lagrangian and these two effective Lagrangians for the case of scattering in a static field $l = l(\mathbf{x})$. In the notation for in- and out states introduced in appendix E, the transition amplitudes $T_\pm$ for particle- and anti-particle scattering are defined by

$$\langle p;\text{out}|q;\text{in}\rangle = \langle p;\text{in}|q;\text{in}\rangle + i2\pi\delta(p^0 - q^0)T_+(p,q) \tag{3.21}$$

$$\langle \bar{p};\text{out}|\bar{q};\text{in}\rangle = \langle \bar{p};\text{in}|\bar{q};\text{in}\rangle + i2\pi\delta(p^0 - q^0)T_-(p,q). \tag{3.22}$$

Fourier transformation is defined as in appendix A with the difference that only the energy is conserved

$$2\pi\delta(p^0 - q^0)T(p,q) = \int d^4x d^4y e^{i(px-qy)}T(x,y). \tag{3.23}$$

The physical momenta of incoming particles (anti-particles) and outgoing particles (anti-particles) are given by $q(-q)$ and $p(-p)$, respectively. Applying the reduction formula of appendix E, we find in the full theory

$$T_\pm(p,q) = T(\pm p, \pm q)|_{p^0=q^0=\omega(\mathbf{p})} , \tag{3.24}$$

whereas the effective theories give

$$T_\pm(p,q) = \frac{1}{\sqrt{2\omega(\mathbf{p})}} \frac{1}{\sqrt{2\omega(\mathbf{q})}} T(\pm p, \pm q)\bigg|_{p^0=q^0=\omega(\mathbf{p})} . \tag{3.25}$$

The additional kinematical factors $1/\sqrt{2\omega}$ are due to the different normalizations of free one-particle states. We have thus verified that the Lagrangians $\mathcal{L}_\pm^{ext}$ produce scattering amplitudes that automatically satisfy the matching condition stated in eq. (2.25).

3.3 Non-local Lagrangians in the Particle and Anti-Particle Sector

We return to the original Lagrangian defined in eq. (3.1), where l represents a dynamical degree of freedom. The results of the last section can be used to construct two non-local Lagrangians that are equivalent to the original theory in the pure particle- and anti-particle sectors of the heavy field including any number of light fields. Ultimately, these Lagrangians will be brought to a local form by expanding in $1/M$. It is the expanded version that is a true *effective* theory in the sense that it reproduces the fundamental theory only at low energies. The non-local version still contains the complete information about truncated Green's functions as we are about to show now.

3.3.1 Green's Functions

We consider the generating functional Z of all Green's functions and perform the integration over the heavy field. In appendix G it is shown that it can be written in the form

$$Z[j, j^*, J] = \frac{1}{\mathcal{Z}} \int [dl] (\det D_+^{-1} \mathcal{D}_+)^{-1} e^{i \int \mathcal{L}_l^0 + j^* D_e^{-1} j + Jl} \tag{3.26}$$

$$\mathcal{Z} = \int [dl] (\det D_+^{-1} \mathcal{D}_+)^{-1} e^{i \int \mathcal{L}_l^0}, \tag{3.27}$$

with D_e and $\mathcal{D}_+$ given in (3.3) and (3.18), respectively. The determinants are evaluated in $D \neq 4$ dimensions where they are finite to all orders in perturbation theory, i.e. we deal here with a regularized but not renormalized theory. The statements derived in this section are a priori only valid within this framework.

In appendix H, we determine the counter terms necessary to render all Green's functions finite in $D=4$ to one loop (i.e. $O(e^2)$). By working only to this order in perturbation theory, the results of this section can be proven to hold also in $D=4$.

Now consider the theory defined by

$$\mathcal{L}_+ = H_+^* \mathcal{D}_+ H_+ + \mathcal{L}_l^0. \tag{3.28}$$

Its generating functional after integration over H_+ is

$$Z_+[j,j^*,J] = \frac{1}{\mathcal{Z}} \int [dl] (\det D_+^{-1} \mathcal{D}_+)^{-1} e^{i\int \mathcal{L}_l^0 - j^* \mathcal{D}_+^{-1} j + Jl}. \tag{3.29}$$

This is simply Z with D_e^{-1} replaced by $-\mathcal{D}_+^{-1}$. In the last section we have found that they can be written as

$$D_e^{-1} = D_M^{-1}\left(1 + T D_M^{-1}\right) \tag{3.30}$$

$$\mathcal{D}_+^{-1} = D_+^{-1}\left(1 - dTdD_+^{-1}\right). \tag{3.31}$$

The first equation is the definition of T which is to be considered as a functional of l within the path integrals above. Let us first consider the n-point functions (the tilde distinguishes them from the connected functions defined below)

$$\begin{aligned}\tilde{G}^{(a,b)}(x,y,z) &= \langle 0 | T\, \hat{H}(x) \hat{H}^+(y) \hat{l}(z) | 0 \rangle \\ &= \frac{1}{i^n} \frac{\delta^n Z}{\widehat{\delta j}^*(x) \widehat{\delta j}(y) \widehat{\delta J}(z)} \bigg|_{j=j^*=J=0},\end{aligned} \tag{3.32}$$

where (a,b) is a pair of integers with $2a+b=n$ and x,y,z are vectors $(x_1,\ldots,x_a)$, $(y_1,\ldots,y_a)$, $(z_1,\ldots,z_b)$. We recall that we use the shorthand notation for the product of fields and the definition of the Fourier transform as given in appendix A. The functions $\tilde{G}_+^{(a,b)}$ of the effective theory are defined through Z_+ in an analogous manner.

The derivatives with respect to the sources j,j^* bring down factors of D_e^{-1} and $\mathcal{D}_+^{-1}$ in Z and Z_+, respectively. It is clear that the free parts D_M^{-1} and D_+^{-1} of eqns. (3.30) and (3.31) only contribute to disconnected Green's functions (except for the two-point functions, see below) and we ignore them for the moment. Denoting a permutation P of the coordinates y_i by

$$P(y_1,\ldots,y_a) = (y_{P_1},\ldots,y_{P_a}),$$

the remaining contributions to $\tilde{G}^{(a,b)}$ and $\tilde{G}_+^{(a,b)}$ can then be written as the sum over all permutations of the term

$$\frac{1}{i^a}\frac{1}{\mathcal{Z}} \int [dl] (\det D_+^{-1}\mathcal{D}_+)^{-1} \prod_{i=1}^{a} f(x_i, y_{P_i}) \prod_{j=1}^{b} l(z_j) e^{i\int \mathcal{L}_l^0}. \tag{3.33}$$

For Z, the function f is given by

$$f(u,v) = \int d^Ds d^Dt \Delta_M(u-s)T(s,t)\Delta_M(t-v) \tag{3.34}$$

and for Z_+ by

$$f(u,v) = -\int d^Ds d^Dt \Delta_+(u-s)d_sT(s,t)d_t\Delta_+(t-v). \tag{3.35}$$

The point is that l only occurs in T, which is the same in both expressions. The free propagators, which form the endpoints of external legs corresponding to heavy particles, and the differential operators d can be taken out of the remaining path integral. Since we have already discarded some disconnected pieces, it is useful to consider only connected Green's functions denoted by $G^{(a,b)}$ and $G_+^{(a,b)}$, generated by the functionals iW and iW_+ defined by

$$e^{iW[j,j^*,J]} \doteq Z[j,j^*,J] \tag{3.36}$$

$$e^{iW_+[j,j^*,J]} \doteq Z_+[j,j^*,J] \tag{3.37}$$

in analogy with eq. (3.32). What we have found above is that these functions differ only by the outermost parts of their external heavy lines. More precisely, if we write (u,v,w are vectors like x,y,z and Δ_m is the propagator of the light field obtained from Δ_M by replacing M by m)

$$G^{(a,b)}(x,y,z) = \frac{1}{i^n}\int d^Dud^Dvd^Dw \prod_{i=1}^a\prod_{j=1}^b \Delta_M(x_i-u_i) \tag{3.38}$$

$$S(u,v,w)\Delta_M(v_i-y_i)\Delta_m(z_j-w_j) \tag{3.39}$$

$$G_+^{(a,b)}(x,y,z) = \frac{(-1)^{2a}}{i^n}\int d^Dud^Dvd^Dw \prod_{i=1}^a\prod_{j=1}^b \Delta_+(x_i-u_i) \tag{3.40}$$

$$d_{u_i}S_+(u,v,w)d_{v_i}\Delta_+(v_i-y_i)\Delta_m(z_j-w_j), \tag{3.41}$$

for $2a+b>2$ we have $S=S_+$ to any order in perturbation theory. In particular, $G_+^{(a,b)}$ has the full loop structure of $G^{(a,b)}$.

Let us consider the two-point functions of the heavy fields in detail. In momentum space we find

$$G^{(1,0)}(p) = \frac{1}{i}\Delta_M(p)\left(1+S(p)\frac{1}{i}\Delta_M(p)\right) \tag{3.42}$$

$$G_+^{(1,0)}(p) = i\Delta_+(p)\left(1+\frac{S(p)}{2\omega(\mathbf{p})}i\Delta_+(p)\right) \tag{3.43}$$

with $S(p)$ being the Fourier transform of

$$S(x-y) = \frac{i}{\mathcal{Z}}\int [dl](\det D_+^{-1}\mathcal{D}_+)^{-1}T(x,y)e^{i\int \mathcal{L}_l^0}. \tag{3.44}$$

The interesting thing about this is that the irreducible two-point functions Σ, Σ_+ defined by

$$G^{(1,0)}(p) = \frac{1}{i} \frac{1}{M^2 - p^2 + i\Sigma(p^2) - i\epsilon} \tag{3.45}$$

$$G_+^{(1,0)}(p) = \frac{1}{i} \frac{1}{\omega(\mathbf{p}) - p^0 + i\Sigma_+(p^0, \mathbf{p}) - i\epsilon} \tag{3.46}$$

automatically obey the equation

$$\Sigma_+(p^0, \mathbf{p}^2) = \frac{\Sigma_(p^2)}{2\omega(\mathbf{p}) + \frac{i\Sigma(p^2)}{\omega(\mathbf{p})+p^0}} \tag{3.47}$$

that was *imposed* as a matching condition in the general discussion of the of two-point functions of a relativistic theory and a non-relativistic effective theory in appendix D. Based on this matching, we have discussed in section 2.2 how off-shell truncated Green's functions can be matched. The statements made there are true in this model and we conclude that if we truncate external lines through the function

$$\mathcal{G}_+(p) \doteq \frac{G_+^{(1,0)}(p)}{\sqrt{2\omega}} \left(1 - \frac{i\Sigma_+(p^0, \mathbf{p}^2)}{\omega(\mathbf{p}) + p^0}\right) \tag{3.48}$$

according to

$$G_+^{(a,b)}(p,q,k) = \hat{\mathcal{G}}_+(p)\hat{\mathcal{G}}_+(q)\hat{G}_+^{(0,2)}(k)\bar{G}_{+,tr}^{(a,b)}(p,q,k), \tag{3.49}$$

the equation

$$G_{tr}^{(a,b)}(p,q,k) = \bar{G}_{+,tr}^{(a,b)}(p,q,k) \tag{3.50}$$

is true to all orders in perturbation theory. Furthermore, the residues Z_H and Z_+ of $G^{(1,0)}$ and $G_+^{(1,0)}$ are related by

$$Z_+(\mathbf{p}^2) = \frac{(\omega(\mathbf{p}) + \omega_{\mathrm{p}}(\mathbf{p}))^2}{4\omega_{\mathrm{p}}(\mathbf{p})\omega(\mathbf{p})} Z_H. \tag{3.51}$$

3.3.2 Amplitudes

As a consequence of eqns. (3.50) and (3.51), the on-shell relation

$$\begin{aligned} &Z_H^a Z_l^{\frac{b}{2}}\ G_{tr}^{(a,b)}(p,q,k)\Big|_{\text{on-shell}} = \\ &\prod_{i=1}^{a} \left(Z_+(\mathbf{p}_i^2)2\omega_{\mathrm{p}}(\mathbf{p}_i)\right)^{\frac{1}{2}} \left(Z_+(\mathbf{q}_i^2)2\omega_{\mathrm{p}}(\mathbf{q}_i)\right)^{\frac{1}{2}} Z_l^{\frac{b}{2}}\ G_{+,tr}^{(a,b)}(p,q,k)\Big|_{\text{on-shell}}, \end{aligned} \tag{3.52}$$

where $p_i^0 = \omega_{\rm p}(\mathbf{p}_i), q_i^0 = \omega_{\rm p}(\mathbf{q}_i)$ and $k_i^0 = \sqrt{m_{\rm p}^2 + \mathbf{k}^2}$ is also true. According to the LSZ formalism, the l.h.s. is related to the amplitude of the process where a heavy particles scatter into a heavy and b light particles[1]

$$\begin{aligned}\langle p_1, \ldots, p_a, k_1, \ldots, k_b; \text{out}|q_1, \ldots, q_a; \text{in}\rangle = \\ \langle p_1, \ldots, p_a, k_1, \ldots, k_b; \text{in}|q_1, \ldots, q_a; \text{in}\rangle \\ + i(2\pi)^4 \delta^4 \left(P + K - Q\right) T_{a\to a+b},\end{aligned} \quad (3.53)$$

where $P = \sum_{i=1}^{a} p_i$ etc. ,through

$$T_{a\to a+b} = \frac{1}{i} Z_H^a Z_l^{\frac{b}{2}}\; G_{tr}^{(a,b)}(p,q,k)\Big|_{\text{on-shell}}. \quad (3.54)$$

The same amplitude in the effective theory is given by

$$T_{a\to a+b}^{+} = \frac{1}{i} \prod_{i=1}^{a} Z_+(\mathbf{p}_i)^{\frac{1}{2}} Z_+(\mathbf{q}_i)^{\frac{1}{2}} Z_l^{\frac{b}{2}}\; G_{+,tr}^{(a,b)}(p,q,k)\Big|_{\text{on-shell}} \quad (3.55)$$

and eq. 3.52 is simply the statement that

$$T_{a\to a+b} = \prod_{i=1}^{a} \sqrt{2\omega_{\rm p}(\mathbf{p}_i)} \sqrt{2\omega_{\rm p}(\mathbf{q}_i)} T_{a\to a+b}^{+}, \quad (3.56)$$

which is nothing but the matching condition stated in section 2.1.4.

We can repeat this procedure with the Lagrangian

$$\mathcal{L}_- = H_-^* \mathcal{D}_- H_- + \mathcal{L}_l^0, \quad (3.57)$$

describing the anti-particle sector of the theory. In the relativistic theory, the amplitude for the process where all particles are replaced by anti-particles is obtained by a simple change of sign of the momenta p and q as a consequence of crossing symmetry. In the effective theory, however, the crossed process is described by its own amplitude $G_{-,tr}^{(a,b)}$ and we get (on-shell has the same meaning as above)

$$\begin{aligned}Z_H^a Z_l^{\frac{b}{2}}\; G_{tr}^{(a,b)}(-p,-q,k)\Big|_{\text{on-shell}} = \\ \prod_{i=1}^{a} \left(Z_-(\mathbf{p}_i^2) 2\omega_{\rm p}(\mathbf{p}_i)\right)^{\frac{1}{2}} \left(Z_-(\mathbf{q}_i^2) 2\omega_{\rm p}(\mathbf{q}_i)\right)^{\frac{1}{2}} Z_l^{\frac{b}{2}}\; G_{-,tr}^{(a,b)}(-p,-q,k)\Big|_{\text{on-shell}}.\end{aligned} \quad (3.58)$$

The connection with the amplitudes $T_{\bar{a}\to\bar{a}+b}$,$T_{\bar{a}\to\bar{a}+b}^{+}$ of the scattering of a anti-particles into a anti-particles and b light particles is analogous to eqns. (3.54) and (3.55) and we arrive at the same conclusions as above.

---

[1]Due to the convention of the Fourier transform given in appendix A the momenta k_i with $k_i^0 = \sqrt{m_{\rm p}^2 + \mathbf{k}^2}$ correspond to outgoing light particles. The amplitude for processes with incoming light particles can be obtained by crossing

We have demonstrated in this section that the non-local Lagrangians $\mathcal{L}_\pm$ defined in eqns. (3.28) and (3.57) generate scattering amplitudes in the pure particle- and anti-particle sector (including any number of light particles) that are related to the corresponding quantities in the full theory by the matching condition described in section 2.1.4. Strictly speaking, the expressions given in eqns. (3.52) and (3.58) are valid to all orders in perturbation theory only in the presence of a regulator that renders all loops finite. However, the non-local theory is related so closely to the original one that it is evident that once the full theory is renormalized to some order in e, these expressions are valid up to the same order, because the very same counter terms render both theories finite at the same time (see appendix H for the explicit renormalization to one loop).

3.3.3 Comment on the Structure of Green's Functions

The seemingly complicated relation (3.50) between the Green's functions of the relativistic and the effective theory is in fact quite simple. Let us illustrate this with the 3-point functions $G^{(1,1)}$ and $G_+^{(1,1)}$ to $O(e^3)$. The former can be depicted as the sum of the graphs[2] of figure 3.1. The corresponding function $G_+^{(1,1)}$ can be

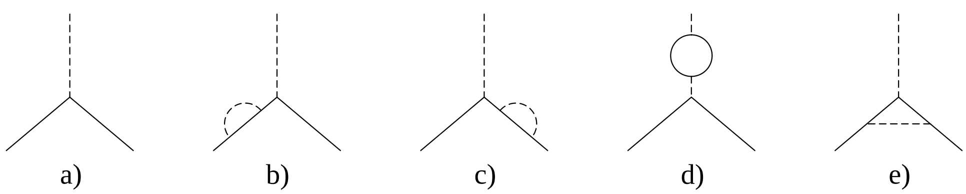


Figure 3.1: The graphs contributing to the 3-point function $G^{(1,1)}$ to $O(e^3)$. The solid and dashed lines represent propagators Δ_M and Δ_m, respectively.

obtained from these graphs by the following simple rules.

- Replace all internal propagators $\Delta_M(p)$ by the sum

$$\Delta_M(p) = -\frac{1}{2\omega(\mathbf{p})}\left(\Delta_+(p) + \Delta_-(p)\right).$$

- Replace all external heavy propagators by particle propagators according to

$$\Delta_M(p) \to \frac{1}{\sqrt{2\omega(\mathbf{p})}}\Delta_+(p).$$

The resulting graphs are shown in figure 3.2. It is convenient to display the decomposition of Δ_M only for the lines that connect 1-particle irreducible subgraphs. The meaning of the truncation rule in eq. (3.49) becomes now apparent. The func-

[2]We omit all tadpole graphs in accordance with the 1-loop renormalization discussed in appendix H

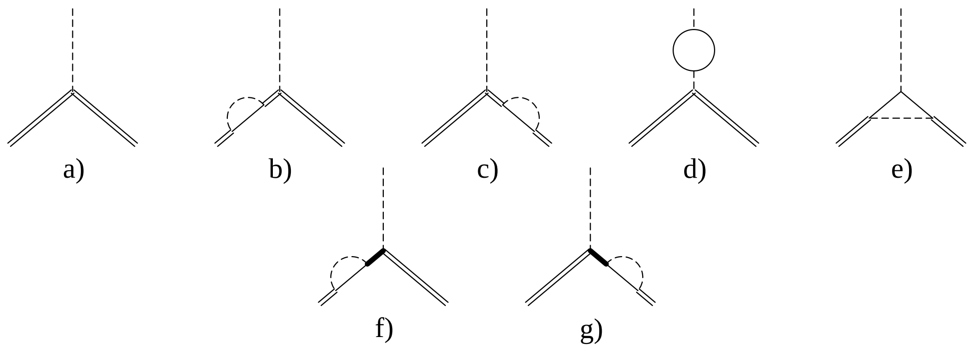


Figure 3.2: The graphs contributing to the 3-point function $G_+^{(1,1)}$ to $O(e^3)$. The solid and dashed lines represent propagators Δ_M and Δ_m, the double line particle propagators $1/(2\omega)\Delta_+$ and the thick solid line anti-particle propagators $1/(2\omega)\Delta_-$. External heavy lines are multiplied with an additional factor of $\sqrt{2\omega}$ so that they effectively correspond to $1/\sqrt{2\omega}\Delta_+$.

tion $G_{tr}^{(1,1)}$ is given by the sum of graphs a) and e) of figure 3.1 with external lines removed. The "naturally" truncated function $G_{+,tr}^{(1,1)}$, however, is the sum of graphs a), e), f) and g) with external factors of Δ_+ and Δ_m removed. The point is that some parts that belong to insertions on the heavy external lines in the relativistic theory are now considered to belong to the irreducible vertex function because the anti-particle propagator Δ_- is considered to be irreducible. The modified truncation rule, involving $\mathcal{G}$ defined in eq. (3.48), on the other hand gives the truncated function $\bar{G}_{+,tr}^{(1,1)}$ which only contains graphs a) and e). We have thus verified explicitly the equation

$$G_{tr}^{(1,1)} = \bar{G}_{+,tr}^{(1,1)}$$

to $O(e^3)$.

Chapter 4

$1/M$ Expansion

The Lagrangians constructed in the preceding chapter are non-local, i.e. they depend on the entire configuration space. The explicit expression for $\mathcal{L}_+$ is

$$\mathcal{L}_+(x) = \int d^4y H_+^*(x) \big(\delta^4(x-y)(D_{+,y} - eB(y)) - \\ e^2 B(x) C^{-1}(x,y) B(y) H_+(y)\big) . \quad (4.1)$$

We showed that this theory contains the same truncated Green's functions as the original local field theory. The whole purpose of the construction of $\mathcal{L}_\pm$ is to pave the way for the expansion of these Green's functions in the region where all energies and momenta are small compared to the mass M. This expansion turns the non-local Lagrangians into local ones, which should be able to reproduce the expansion of relativistic Green's functions.

In this chapter, we first look at a few simple processes in the relativistic theory and discuss their $1/M$ expansion at tree level. Then we perform the expansion in the non-local Lagrangian and discuss how perturbation theory works. Finally, we check the method in the case of the scattering of a heavy and a light particle at tree level.

4.1 Expansion of Relativistic Amplitudes at Tree Level

We consider the truncated Green's functions $G_{tr}^{(2,0)}$ and $G_{tr}^{(1,2)}$ on the mass shell, i.e. the heavy momenta obey $p^2 = M_\mathrm{p}^2$ and the light momenta $k^2 = m_\mathrm{p}^2$.

4.1.1 Heavy-Heavy Scattering

The function $G_{tr}^{(2,0)}(p_1, p_2, q_1, q_2)$ involves only heavy external particles. With the convention for the Fourier transform of Green's functions given in appendix A, q_1,q_2 are the physical momenta of incoming particles and p_1,p_2 those of outgoing ones. Therefore, $q_1 + q_2$ is the total energy in the CMS of particle-particle

scattering. We define the Mandelstam variables

$$\begin{aligned} s &= (q_1 + q_2)^2 \\ t &= (q_1 - p_1)^2 \\ u &= (q_1 - p_2)^2 \end{aligned} \tag{4.2}$$

related by

$$s + t + u = 4M_{\mathrm{p}}^2. \tag{4.3}$$

The invariant amplitude

$$A(s,t,u) = \frac{1}{i} Z_H^2 \, G_{tr}^{(2,0)}(p_1,p_2,q_1,q_2)\Big|_{\text{on-shell}} \tag{4.4}$$

describes several physical processes in different regions of momentum space (cf. figure 4.1). We define the amplitudes belonging to the various channels by

$$\begin{aligned} A_s(s,t,u) &= A(s,t,u)|_{q_1^0,q_2^0,p_1^0,p_2^0>0} \\ A_t(t,s,u) &= A(s,t,u)|_{q_1^0,p_2^0>0;q_2^0,p_1^0<0} \\ A_u(u,t,s) &= A(s,t,u)|_{q_1^0,p_1^0>0;q_2^0,p_2^0<0}\,, \end{aligned} \tag{4.5}$$

writing the energy in the CMS and the momentum transfer as the first and second arguments, respectively. In the s-channel, $A(s,t,u)$ describes particle-particle scattering and in the t- and u-channels particle-anti-particle scattering. The pres-

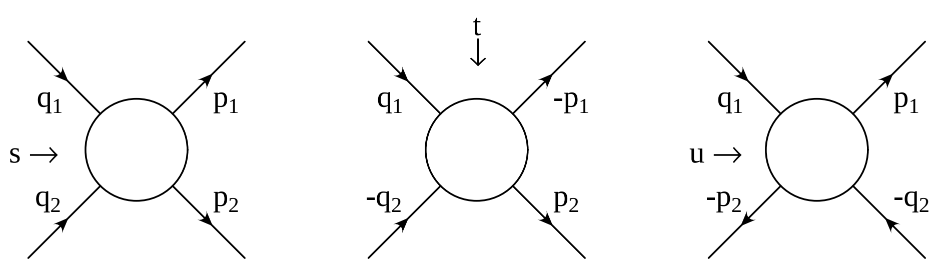


Figure 4.1: Physical processes associated with the amplitude $A(s,t,u)$ defined in eq. (4.4). In the s-channel it describes the scattering of two heavy particles and in the t- and u-channels the scattering of a particle and an anti-particle. The lines are labeled by the physical momenta in the respective channels.

ence of identical particles is reflected in the crossing symmetry

$$A(s,t,u) = A(s,u,t). \tag{4.6}$$

In perturbation theory we write

$$A(s,t,u) = e^2 A^{(2)}(s,t,u) + O(e^4) \tag{4.7}$$

and find that the lowest order is given by the two tree-level Feynman diagrams shown in figure 4.2

$$A^{(2)}(s,t,u) = \frac{1}{m^2 - t} + \frac{1}{m^2 - u}. \tag{4.8}$$

Let us expand this quantity for the case when all three-momenta as well as m

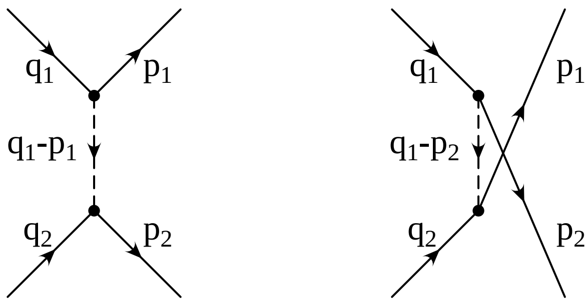


Figure 4.2: The graphs that contribute to $A(s,t,u)$ at tree-level.

are much smaller than the heavy scale M. This expansion has to be performed separately in each channel and we start with the s-channel. It is convenient to work in the CMS, where $q_1 = (\sqrt{s}/2, \mathbf{q})$, $q_2 = (\sqrt{s}/2, -\mathbf{q})$, $p_1 = (\sqrt{s}/2, \mathbf{p})$ and $p_2 = (\sqrt{s}/2, -\mathbf{p})$. The CM energy s is of the order of $4M^2$ and thus represents a hard scale, where as the momentum transfer t and $u = 4M^2 - s - t$ are soft

$$t = -(\mathbf{q} - \mathbf{p})^2 \tag{4.9}$$

$$u = -(\mathbf{q} + \mathbf{p})^2 \tag{4.10}$$

Therefore, both denominators in eq. (4.8) are small and

$$A_s^{(2)}(s,t,u) = \frac{1}{m^2 + (\mathbf{q} - \mathbf{p})^2} + \frac{1}{m^2 + (\mathbf{q} + \mathbf{p})^2}. \tag{4.11}$$

In the t-channel, t is the hard CM energy. In the CMS, where $q_1 = (\sqrt{t}/2, \mathbf{q})$, $p_1 = (-\sqrt{t}/2, \mathbf{q})$, $q_2 = (-\sqrt{t}/2, \mathbf{p})$ and $p_2 = (\sqrt{t}/2, \mathbf{p})$, we have

$$t = 4(M^2 + \mathbf{q}^2). \tag{4.12}$$

The momentum transfer s and

$$u = -(\mathbf{q} - \mathbf{p})^2 \tag{4.13}$$

are still soft. In this channel, the first graph of figure 4.2 represents an annihilation process, where the particle and anti-particle convert into a light particle which is then considerably off its mass shell, followed by pair production. The leading term of the expanded propagator is of $O(1/M^2)$ and indicates that this process looks

essentially point-like in configuration space on a scale much larger than $1/M$. The second term involves only the exchange of soft momenta and has a leading piece that is not suppressed by powers of $1/M$

$$A_t^{(2)}(t,s,u) = \frac{1}{m^2 + (\mathbf{q}-\mathbf{p})^2} - \frac{1}{4M^2}\left(1 - \frac{4\mathbf{q}^2 - m^2}{4M^2} + O(\frac{1}{M^4})\right). \quad (4.14)$$

4.1.2 Heavy-Light Scattering

Let us chose the momentum assignment in the Fourier transform of $G^{(1,2)}$ as follows

$$(2\pi)^4\delta^4(p+k_2-q-k_1)G^{(1,2)}(p,q,k_1,k_2) = \int d^4x d^4y d^4z_1 d^4z_2 e^{ipx-iqy+ik_2z_2-ik_1z_1} G^{(1,2)}(x,y,z_1,z_2). \quad (4.15)$$

With this choice, q,k_1 are the physical momenta of incoming particles and p,k_2 those of outgoing ones. Therefore, $q+k_1$ is the total energy in the CMS of the process where a light particle scatters off a heavy one and we chose the Mandelstam variables

$$\begin{aligned} s &= (q+k_1)^2 \\ t &= (q-p)^2 \\ u &= (q-k_2)^2 \end{aligned} \quad (4.16)$$

with

$$s+t+u = 2(M_\mathrm{p}^2 + m_\mathrm{p}^2). \quad (4.17)$$

The different processes represented by

$$B(s,t,u) = \frac{1}{i} Z_H Z_l\, G_{tr}^{(1,2)}(p,q,k_1,k_2)\Big|_{\text{on-shell}} \quad (4.18)$$

are shown in figure 4.3 and the amplitudes in the different channels are defined in analogy with eq. (4.5)

$$\begin{aligned} B_s(s,t,u) &= B(s,t,u)|_{q^0,p^0,k_1^0,k_2^0>0} \\ B_t(t,s,u) &= B(s,t,u)|_{q^0,k_2^0>0;p^0,k_1^0<0} \\ B_u(u,t,s) &= B(s,t,u)|_{q^0,p^0>0;k_1^0,k_2^0<0}\,. \end{aligned} \quad (4.19)$$

We refer to the s- and u-channels as Compton scattering and the t-channel as pair-annihilation. Because of the crossing symmetry

$$B(s,t,u) = B(u,t,s), \quad (4.20)$$

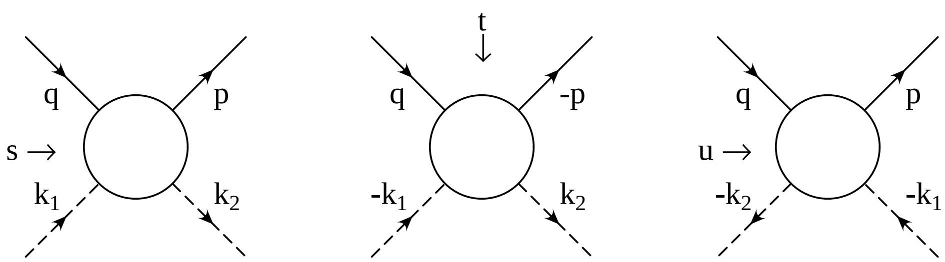


Figure 4.3: Physical processes associated with the amplitude $B(s,t,u)$ defined in eq. (4.18). Solid and dashed lines represent heavy and light particles, respectively. In the s- and u- channels it describes scattering and in the t-channel pair-annihilation. The lines are labeled by the physical momenta in the respective channels.

we can again restrict the analysis to the s- and t-channels. Let us set

$$B(s,t,u) = e^2 B^{(2)}(s,t,u) + O(e^4) \tag{4.21}$$

where $B^{(2)}$ is given by the Feynman diagrams displayed in figure 4.4

$$B^{(2)}(s,t,u) = \frac{1}{M^2 - s} + \frac{1}{M^2 - u}. \tag{4.22}$$

In contrast to the processes considered above, this amplitude explicitly depends

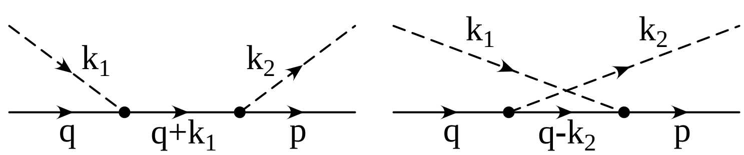


Figure 4.4: The graphs that contribute to $B(s,t,u)$ at tree-level.

on the heavy scale through the propagator

$$\Delta_M(p) = \frac{1}{M^2 - p^2}. \tag{4.23}$$

The construction of the non-local Lagrangians $\mathcal{L}_\pm$ relied essentially on the decomposition

$$\Delta_M(p) = \frac{1}{2\omega(\mathbf{p})} \left(\frac{1}{\omega(\mathbf{p}) - p^0} + \frac{1}{\omega(\mathbf{p}) + p^0} \right). \tag{4.24}$$

of this function, representing the propagation of a particle and an anti-particle separately. The important point is that when p^0 is in the vicinity of $+\omega(\mathbf{p})$, the

first term dominates where as the second one can be expanded in powers of $1/M$ and vice versa if p^0 is in the vicinity of $-\omega(\mathbf{p})$. In configuration space, the first graph of figure 4.4 may be depicted as the sum of the two graphs in figure 4.5. In

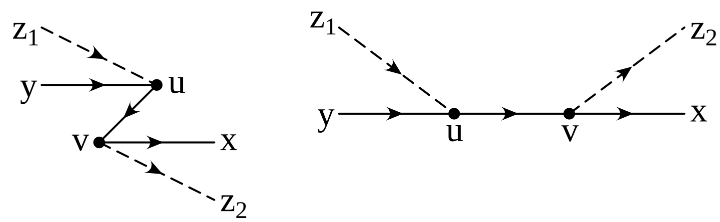
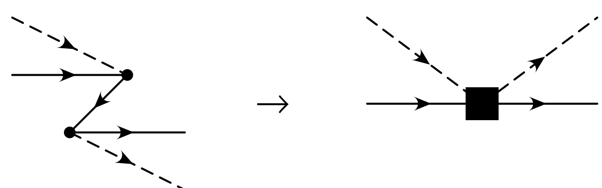


Figure 4.5: Decomposition of the first graph of figure 4.4 according to eq. (4.24) in configuration space (an integration over the internal points u,v is implied).

these diagrams, the internal propagators correspond to factors $d^2\Delta_-(v-u)$ and $d^2\Delta_+(u-v)$, respectively (cf. appendix B for the definition of these objects). Due to its shape, the first graph is called a “Z” graph in the language of old-fashioned (non-covariant) perturbation theory.

In the s-channel, the incident light particle pushes the incident heavy particle only slightly off the mass shell, so that the internal anti-particle propagator in the Z graph is far away from its pole at $p^0 = -\omega(\mathbf{p})$ and is suppressed relative to the other graph. The Z graph looks like an effective local four-particle interaction



Let us work in the rest frame of the incoming heavy particle where $q = (M, 0)$, $k_1 = (\Omega(\mathbf{k}_1), \mathbf{k}_1)$, $p = (\omega(\mathbf{p}), \mathbf{p})$, $k_2 = (\Omega(\mathbf{k}_2), \mathbf{k}_2)$ and $\Omega(\mathbf{k}) = \sqrt{m^2 + \mathbf{k}^2}$. The contribution to the amplitude $B_s^{(2)}(s,t,u)$ of the Z graph is

$$\frac{1}{2\omega(\mathbf{k}_1)} \frac{1}{\omega(\mathbf{k}_1) + \Omega(\mathbf{k}_1) + M} = \frac{1}{4M^2}\left(1 - \frac{\Omega(\mathbf{k}_1)}{2M} + O(\frac{1}{M^2})\right). \quad (4.25)$$

The other part of the diagram gives the leading contribution

$$\frac{1}{2\omega(\mathbf{k}_1)} \frac{1}{\omega(\mathbf{k}_1) - \Omega(\mathbf{k}_1) - M} = \frac{-1}{2M\Omega(\mathbf{k}_1)}\left(1 + \frac{\mathbf{k}_1^2}{2M\Omega(\mathbf{k}_1)} + O(\frac{1}{M^2})\right) \quad (4.26)$$

and

$$B_s^{(2)}(s,t,u) = \frac{-1}{2M\Omega(\mathbf{k}_1)}\left(1+\frac{\mathbf{k}_1^2}{2M\Omega(\mathbf{k}_1)}\right) + \frac{1}{4M^2}\left(1-\frac{\Omega(\mathbf{k}_1)}{2M}\right)$$
$$+ (\mathbf{k}_1 \to -\mathbf{k}_2, \Omega(\mathbf{k}_1) \to -\Omega(\mathbf{k}_2)) + O(\frac{1}{M^2}). \qquad (4.27)$$

In the t-channel, things are different again. Let us chose the CMS and go to the threshold, where $q = (M, 0)$, $p = (-M, 0)$, $k_1 = (-M, \mathbf{k})$ and $k_2 = (M, \mathbf{k})$ (remember that the physical momenta are q, $-p$, $-k_1$ and k_2). The invariants have the values $s = u = m^2 - M^2$ and

$$B_t(u,t,s) = \frac{2}{2M^2 - m^2} = \frac{1}{M^2}\left(1 + \frac{m^2}{2M^2} + O(\frac{1}{M^4})\right). \qquad (4.28)$$

This means that pair-annihilation has no soft component: the entire process looks local on a scale much larger than $1/M$.

To summarize, we may group all processes we have just discussed into three categories. If the initial and final states contain exclusively either heavy particles or anti-particles, we call it a *soft* process (charge conservation implies that the number of particles is conserved). If the initial and final states contain both types of particles but their number is separately conserved, we call it a *semi-hard* process. Finally, if the numbers of particles and anti-particles are not conserved separately, we call it a *hard* process. The number of light particles is not important for this classification.

- Soft processes. This category comprises the s-channel of the amplitude A and the s- and u-channels of the amplitude B (i.e. particle-particle and Compton scattering). They have in common that at each vertex of the tree-level diagrams, only energies and momenta that are much smaller than M are transferred. This means that all virtual light particles are not far from the mass shell, mediating the interaction over distances that are not small compared to $1/M$, and all virtual heavy particles are in the vicinity of the particle mass shell, i.e. the energy component of its momentum is close to ω. Therefore, only the anti-particle components of these propagators represent a local interaction. As a consequence, no more than two heavy lines are attached to a local effective vertex.

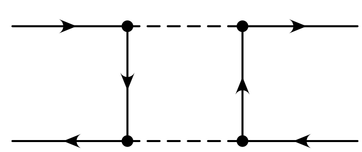
- Semi-hard processes. The t- and u channels, describing particle-anti-particle scattering, of the amplitude A are the only members of this category. In the annihilation channel (the first graph of figure 4.2 in the t-channel and the second graph in the u-channel) a heavy particle annihilates with a heavy anti-particle, emitting a virtual light particle that is well off its mass shell and travels only a distance of the order of $1/M$, giving rise to local interactions with more than two heavy particles involved. The other contribution to the process is soft in the sense described above.

- Hard processes. The pair-annihilation (the t-channel of the amplitude B) is completely local because the virtual particles are always far away from the mass shell. This can be traced to the fact that at least one of the emerging light particles must be hard: even at threshold, the energy released by the annihilating particles is of the order of M.

Let us discuss the hard processes in more detail. In the terminology just established, the process where two heavy particles annihilate into, say, 100 light particles is still considered to be hard. One may think that this is not adequate, because each of the light particles can be very soft. However, there are still some regions of phase space where a sizeable fraction of the energy is distributed among a few of them, which are then hard. Thus, the expansion of internal heavy lines depends on the configuration of the final states and it seems that there is no expansion that is valid everywhere in phase space. One may say that some pieces of the amplitude require one to treat both, particle and anti-particle as heavy degrees of freedom. The point is that neither $\mathcal{L}_+$ nor $\mathcal{L}_-$ are valid in this region.

Looking at equation (4.28), one might be tempted to simply add a local interaction of the type $H_+H_-^*l^2$ (and its hermitian conjugate), since the entire process is local. Such a term contributes also to the two-point function $\langle 0|TH_+(x)H_+^\dagger(y)|0\rangle$ at $O(e^4)$. Now, this Green's function is already correctly described by $\mathcal{L}_+$ alone, as we have seen in section 3.3, and there arises the problem of double counting: by adding the mentioned local term, we must change the coefficients of $\mathcal{L}_+$ already fixed by a matching in the particle sector. It is a priori not clear if this procedure can be implemented systematically.

In addition, unitarity tells us that the tree-level amplitudes of figure 4.4 in the t channel are related to the imaginary part of the diagram



in the particle-anti-particle channel. Being a semi-hard process, we expect that the box is represented as a string of local four-particle (two particle and two anti-particle) interactions. This is again in conflict with a term of the form $H_+H_-^*l^2$, because two of these vertices essentially generate the box itself.

These are the reasons why annihilation processes are usually excluded from the effective Lagrangian. Attempts have been made to include them in order to describe positronium decay [3] or heavy quarkonium decay [26].

Clearly, this subject deserves further investigation.

4.2 Off-Shell Expansion

In section 3.3 we have seen that we can reproduce the truncated off-shell Green's functions of the relativistic theory if we use a special truncation prescription in the

effective theory, which amounts to multiply with an additional factor $\sqrt{2\omega}$ for every external heavy line at tree level. The local effective theory is expected to produce a $1/M$ expansion of Green's functions. Let us therefore extend the expansion discussed in the last section to off-shell momenta, i.e. we go back to the functions $G_{tr}^{(2,0)}$ and $G_{tr}^{(1,2)}$, treating the energy components of the momenta as independent variables.

4.2.1 Heavy-Heavy scattering

We keep the notation with Mandelstam variables but discard the on-shell conditions. Strictly speaking, we cannot talk about different channels any more because we are outside of the physical region. However, to stay in the scope of a $1/M$ expansion, we cannot move too far away from the mass shell so that the notion of channels still has some meaning. In the s-channel, for example, we restrict the energies of the particles to be much smaller than M in the sense that $|q_i^0 - M|$, $|p_1^0 - M| \ll M$. It is convenient to introduce new variables ($i = 1, 2$)

$$E_{q_i} \doteq q_i^0 - M \tag{4.29}$$

$$E_{p_i} \doteq p_i^0 - M. \tag{4.30}$$

The Green's function depends on several small dimensionless quantities E_{q_i}/M, $|\mathbf{q}_i|/M, \ldots$ and we must decide what their relative magnitude is. At the moment, we do not have any preference and simply consider all of them to be of equal magnitude, which is the same as counting powers of $1/M$ as before. More about this issue will be said below. In this framework, the function

$$\begin{aligned}\frac{1}{i}G_{tr}^{(2,0)}(p_1,p_2,q_1,q_2) = &\frac{1}{m^2 - (E_{q_1} - E_{p_1})^2 + (\mathbf{q}_1 - \mathbf{p}_1)^2} \\ &+ \frac{1}{m^2 - (E_{q_1} - E_{p_2})^2 + (\mathbf{q}_1 - \mathbf{p}_2)^2}\end{aligned} \tag{4.31}$$

cannot be expanded at all.

In the t-channel, $|p_1^0 - M|$ and $|q_2^0 - M|$ are of the order of M. The good variables are in this case

$$\bar{E}_{p_1} \doteq p_1^0 + M \tag{4.32}$$

$$\bar{E}_{q_2} \doteq q_2^0 + M \tag{4.33}$$

in the sense that $|\bar{E}_{p_1}|$, $|\bar{E}_{q_2}| \ll M$. We find

$$\begin{aligned}\frac{1}{i}G_{tr}^{(2,0)}(p_1,p_2,q_1,q_2) = &\frac{1}{m^2 - (E_{q_1} - E_{p_2})^2 + (\mathbf{q}_1 - \mathbf{p}_2)^2} \\ -\frac{1}{4M^2}\left(1 - \frac{E_{q_1} - \bar{E}_{p_1}}{M} + \frac{3(E_{q_1} - \bar{E}_{p_1})^2 + (\mathbf{q}_1 - \mathbf{p}_1)^2 + m^2}{4M^2} + O(\frac{1}{M^3})\right).\end{aligned} \tag{4.34}$$

4.2.2 Heavy-Light scattering

Using the same energy variables as before and considering the energy components of the light momenta to be of the same order, we find in the s-channel

$$\begin{aligned}\frac{1}{i}G_{tr}^{(1,2)}(p,q,k_1,k_2) = \frac{-1}{2M(E_q+k_1^0)}\left(1+\frac{(\mathbf{q}+\mathbf{k}_1)^2}{2M(E_q+k_1^0)}+\right.\\ \left.+\frac{(\mathbf{q}+\mathbf{k}_1)^4}{4M^2(E_q+k_1^0)^2}-\frac{(\mathbf{q}+\mathbf{k}_1)^2}{4M^2}O(\frac{1}{M^3})\right)\\ +\frac{1}{4M^2}\left(1-\frac{E_q+k_1^0}{2M}+O(\frac{1}{M^2})\right)+(k_1\to -k_2). \quad (4.35)\end{aligned}$$

4.3 Effective Local Lagrangians for Soft Processes

The non-local theories constructed in section 3.3 are naturally restricted to soft processes in the particle and anti-particle sectors and we have proven that they reproduce the relativistic theory exactly at tree level. The effective local Lagrangians are obtained by expanding the non-local pieces, which are the anti-particle propagator Δ_- and the particle propagator Δ_+ for $\mathcal{L}_+$ and $\mathcal{L}_-$, respectively. Let us first concentrate on $\mathcal{L}_+$. We find

$$\begin{aligned}\Delta_-(x) = & -\frac{1}{2M}\left(1-\frac{i\partial_t-M}{2M}+\frac{\Delta}{4M^2}\right.\\ & \left.+\frac{(i\partial_t-M)^2}{4M^2}+O(\frac{1}{M^3})\right)\delta^4(x) \quad (4.36)\end{aligned}$$

and, expanding the operator $d = (2\sqrt{M^2-\Delta})^{-1/2}$ as well, we can write the Lagrangian in the form

$$\mathcal{L}_+ = H_+^* D_+ H_+ + \sum_{n=1}^{\infty}\frac{1}{(2M)^n}\mathcal{L}_+^{(n)}, \quad (4.37)$$

where

$$\mathcal{L}_+^{(1)} = eH_+^* l H_+ \quad (4.38)$$

$$\mathcal{L}_+^{(2)} = 0 \quad (4.39)$$

$$\mathcal{L}_+^{(3)} = H_+^*\left\{e(l\Delta+\Delta l)+e^2l^2\right\}H_+ \quad (4.40)$$

$$\mathcal{L}_+^{(4)} = -e^2H_+^* l(i\partial_t-M)lH_+ \quad (4.41)$$

$$\begin{aligned}\mathcal{L}_+^{(5)} = H_+^*\Big\{ & e\left(\Delta l\Delta+\frac{5}{2}l\Delta^2+\frac{5}{2}\Delta^2 l\right)\\ & +e^2\left(l^2\Delta+\Delta l^2+3l\Delta l+l[i\partial_t-M]^2l\right)\\ & +e^3l^3\Big\}H_+ \quad (4.42)\end{aligned}$$

and the differential operators act on everything on their right. In the anti-particle sector, the Lagrangian is of the same form and the $\mathcal{L}_-^{(n)}$ are obtained from the $\mathcal{L}_+^{(n)}$ by replacing H_+ by H_- and $i\partial_t - M$ by $-i\partial_t - M$.

4.3.1 Including Semi-Hard Processes

The semi-hard processes contain virtual pair-annihilation and creation processes, represented by local effective interactions of several heavy particles. It is clear that a candidate for the effective theory that should include these reactions must contain both types of heavy particles. Consider the Lagrangian

$$\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_-. \tag{4.43}$$

It clearly contains the pure particle- and anti-particle sectors as well as the soft part of the semi-hard particle-anti-particle processes but not the hard part of the latter. To include those, we must supplement the Lagrangian with contact interactions between particles and anti-particles of the form

$$\mathcal{L}_c = \sum_{n=1}^{\infty} e^{2n} \mathcal{L}_c^{(n)}, \tag{4.44}$$

where $\mathcal{L}_c^{(n)}$ contains n factors of the fields H_+,H_+^*,H_- and H_-^*. Each of these terms is itself an expansion in $1/M$

$$\mathcal{L}_c^{(n)} = \sum_{m=0}^{\infty} \frac{1}{(2M)^{4n+m}} \mathcal{L}_c^{(n,m)}. \tag{4.45}$$

The first two terms of $\mathcal{L}_c^{(1)}$ can be read off from the second term of eq. (4.34)

$$\mathcal{L}_c^{(1,0)}(x) = H_+^*(x) H_+(x) H_-^*(x) H_-(x) \tag{4.46}$$

$$\begin{aligned}\mathcal{L}_c^{(1,1)}(x) = &-2H_+^*(x)\left([(i\partial_{x^0} - M)H_+(x)]H_-^*(x)\right.\\ &\left.+H_+(x)[(i\partial_{x^0} - M)H_-^*(x)]\right) H_-(x).\end{aligned} \tag{4.47}$$

4.4 Power Counting Schemes

In the relativistic theory, there is only one expansion parameter: the coupling constant e. The effective theory contains many more small parameters, namely the energies and momenta of the process of interest, which are considered to be small compared to M. In such a multiple expansion, the question of ordering arises, i.e. what is the relative magnitude of the expansion parameters, which determines what terms in the expansion should be grouped together. We refer to a particular ordering as a *power counting scheme*. In the following we discuss the two schemes which are of practical importance.

Because the effective theory should reproduce quantities of the fundamental theory, the primary expansion parameter is the coupling e. If we go to the mass shell, the energies of the particles are expressed in terms of their momenta and the number of independent expansion parameters is reduced. In section 4.1, we have expanded some on-shell amplitudes to a fixed order in e and some power of $1/M$, i.e. we have collected terms with the same powers of e and $1/M$. Formally, we may introduce a small number v as a bookkeeping device and assign powers of it to the expansion parameters after making them dimensionless by dividing with appropriate powers of M. To the momentum $\mathbf{p}$ of a heavy or a light particle we assign

$$\frac{|\mathbf{p}|}{M} = O(v). \tag{4.48}$$

The energy $\Omega(\mathbf{k}) = \sqrt{m^2 + \mathbf{k}^2}$ of a light particle is counted as

$$\frac{\Omega(\mathbf{k})}{M} = O(v). \tag{4.49}$$

This implies that, formally, m/M is considered to be of the same order as $|\mathbf{k}|/M$. As a consequence of these assignments, $|\mathbf{p}|/\Omega(\mathbf{k})$ is of order one. In this language, we would say, for example, that the amplitude $B_s^{(2)}$ in eq. (4.27) is correct up to terms of $O(v^2)$.

In the off-shell expansion performed in section 4.2, we have simply counted powers of $1/M$. This is equivalent to setting

$$\frac{|E|}{M} = O(v) \qquad\qquad \frac{|k^0|}{M} = O(v), \tag{4.50}$$

where E is the energy component of the four vector of a heavy particle with the mass M subtracted and k^0 the energy of a light particle. Clearly, the assignment of E/M is not compatible with the one of $|\mathbf{p}|/M$ if we go on-shell, because

$$E = p^0 - M = \frac{\mathbf{p}^2}{2M} + O(Mv^4), \tag{4.51}$$

i.e. E becomes a quantity of $O(v^2)$. However, no harm is done, because we formally consider E to be larger than it actually is on-shell. This can be seen, for example, in the amplitude $A_t^{(2)}$. The $1/M$ suppressed contribution to the on-shell function is given by[1] (eq. (4.14))

$$-\frac{1}{4M^2}\left(1 - \frac{4\mathbf{q}^2 - m^2}{4M^2} + O(v^4)\right), \tag{4.52}$$

[1] In the CMS

where as the off shell expansion yields[2] (eq. (4.34))

$$-\frac{1}{4M^2}\left(1-\frac{E_{q_1}-\bar{E}_{p_1}}{M}+\frac{3(E_{q_1}-\bar{E}_{p_1})^2+(\mathbf{q}_1-\mathbf{p}_1)^2+m^2}{4M^2}+O(v^3)\right). \tag{4.53}$$

On-shell we have $E_{q_1}=\frac{\mathbf{q}_1^2}{2M}+O(Mv^4)$ and $\bar{E}_{p_1}=-\frac{\mathbf{p}_1^2}{2M}+O(Mv^4)$ and, going to the CMS, both expressions agree to $O(v^2)$.

Of course, we could just as well have performed the off-shell expansion by setting

$$\frac{|E|}{M}=O(v^2). \tag{4.54}$$

In this case, we get

$$-\frac{1}{4M^2}\left(1-\frac{4M(E_{q_1}-\bar{E}_{p_1})-(\mathbf{q}_1-\mathbf{p}_1)^2-m^2}{4M^2}+O(v^4)\right), \tag{4.55}$$

which also agrees with the previous expressions on-shell and to $O(v^2)$.

From this discussion, we can learn two things

- The expansion of on-shell amplitudes is naturally associated with an expansion in $1/M$.
- There is no natural choice for the expansion of off-shell amplitudes (or Green's functions). Counting E/M the same as $|\mathbf{p}|/M$ conserves the strict $1/M$ expansion but the orders get mixed if we go on-shell (the terms E/M will contribute to all higher orders). If we count E/M as $|\mathbf{p}|^2/M^2$, we do not expand simply in $1/M$ but the energies are considered to be of the order they actually are on-shell. Also in this case does a term E/M contribute to all higher orders if we go on-shell. Different counting schemes are possible but not of practical importance.

To conclude, we define two power counting schemes (p and k denote the four-momenta of a heavy and a light particle, respectively):

1. Heavy-Meson (HM) scheme. Its defining feature is that the three-momentum and the energy variable (with the heavy mass subtracted) are considered to be of equal magnitude

$$\frac{|E|}{M}=O(v) \qquad \frac{|k^0|}{M}=O(v) \tag{4.56}$$
$$\frac{|\mathbf{p}|}{M}=O(v) \qquad \frac{|\mathbf{k}|}{M}=O(v). \tag{4.57}$$

   The name is an adaptation from HBCHPT [22], where this counting scheme is used.

[2] In no particular frame of reference

2. Non-Relativistic (NR) scheme. In this scheme, the energy of a heavy particle is counted like its three-momentum squared

$$\frac{|E|}{M} = O(v^2) \qquad \frac{|k^0|}{M} = O(v) \tag{4.58}$$

$$\frac{|\mathbf{p}|}{M} = O(v) \qquad \frac{|\mathbf{k}|}{M} = O(v). \tag{4.59}$$

The name is derived from the fact that the lowest order effective Lagrangian is Galilei-invariant and thus represents a true non-relativistic theory.

4.5 Perturbation Theory

We have seen that the relativistic Green's functions can be expanded in different ways. What does this mean for the effective theory? The effective Lagrangian contains space and time derivatives of the fields. In momentum space they become three-momentum and energy variables and the question of ordering arises already on the level of the Lagrangian. It is clear that the perturbation theory looks different for the different counting schemes. The resulting Green's functions can then be identified with the different expansions of the relativistic Green's functions.

In section 4.3, we have ordered the effective Lagrangian according to powers of $1/M$. It is useful to reorder it now. First of all, we should collect the terms with the same power of e or, which is equivalent, the same number of light fields. Then we should assign the differential operators ∂_t/M and ∇/M some power of the parameter v introduced in the previous section according to one of the power counting schemes. The Lagrangian is then of the form

$$\mathcal{L}_+ = \bar{\mathcal{L}}_+^0 + \mathcal{L}_l^0 + \sum_{\mu,\nu}^{\infty} \left(\frac{e}{2M}\right)^{\mu} \mathcal{L}_+^{(\mu,\nu)}. \tag{4.60}$$

Here, $\bar{\mathcal{L}}_+^0$ contains only the leading part of $\mathcal{L}_+^0$ in the parameter v. The term $\mathcal{L}_+^{(\mu,\nu)}$ contains μ light fields and ν powers of v.

In the HM scheme we formally assign

$$\frac{i\partial_t - M}{M} = O(v) \tag{4.61}$$

$$\frac{\nabla}{M} = O(v) \tag{4.62}$$

irrespective of what field they act on. The interaction independent pieces are given by

$$\bar{\mathcal{L}}_+^0 \equiv \mathcal{L}_{+,\mathrm{HM}}^0 \doteq H_+^* D_{+,\mathrm{HM}} H_+ \tag{4.63}$$

$$D_{+,\mathrm{HM}} = i\partial_t - M \tag{4.64}$$

$$\sum_{m=2}^{\infty} \mathcal{L}_+^{(0,m)} = H_+^*(M - \sqrt{M^2 - \Delta})H_+. \tag{4.65}$$

In the NR scheme we count time derivatives differently:

$$\frac{i\partial_t - M}{M} = O(v^2) \text{ when acting on a heavy field} \tag{4.66}$$

$$\frac{i\partial_t}{M} = O(v) \text{ when acting on a light field} \tag{4.67}$$

$$\frac{\nabla}{M} = O(v) \text{ always} \tag{4.68}$$

and

$$\bar{\mathcal{L}}^0_+ \equiv \mathcal{L}^0_{+,\text{NR}} \doteq H^*_+ D_{+,\text{NR}} H_+ \tag{4.69}$$

$$D_{+,\text{NR}} = i\partial_t - M + \frac{\Delta}{2M} \tag{4.70}$$

$$\sum_{m=2}^{\infty} \mathcal{L}^{(0,m)}_+ = H^*_+ (M - \frac{\Delta}{2M} - \sqrt{M^2 - \Delta}) H_+. \tag{4.71}$$

The leading terms of the interaction Lagrangians $\mathcal{L}^{(\mu,\nu)}_+$ for both schemes are shown in table 4.1.

(μ,ν)	HM	NR
$(1,0)$	$H^*_+ l H_+$	$H^*_+ l H_+$
$(1,2)$	$\frac{1}{4M^2} H^*_+ (l\Delta + \Delta l) H_+$	$\frac{1}{4M^2} H^*_+ (l\Delta + \Delta l) H_+$
$(1,4)$	$\frac{1}{16M^4} H^*_+ (\Delta l \Delta + \frac{5}{2} l \Delta^2$ $+\frac{5}{2}\Delta^2 l) H_+$	$\frac{1}{16M^4} H^*_+ (\Delta l \Delta + \frac{5}{2} l \Delta^2$ $+\frac{5}{2}\Delta^2 l) H_+$
$(2,0)$	$\frac{1}{2M} H^*_+ l^2 H_+$	$\frac{1}{2M} H^*_+ l^2 H_+$
$(2,1)$	$-\frac{1}{4M^2} H^*_+ l (i\partial_t - M) l H_+$	$\frac{-1}{4M^2} H^*_+ l (i\partial_t l) H_+$
$(2,2)$	$\frac{1}{8M^3} H^*_+ (l^2\Delta + \Delta l^2]$ $+3l\Delta l + l[i\partial_t - M]^2 l) H_+$	$\frac{1}{8M^3} H^*_+ (l^2\Delta + \Delta l^2 + 3l\Delta l$ $-l(\partial_t^2 l) - \frac{1}{2M} l^2 (i\partial_t - M)) H_+$
$(3,0)$	$\frac{1}{4M^2} H^*_+ l^3 H_+$	$\frac{1}{4M^2} H^*_+ l^3 H_+$

Table 4.1: The leading terms of the interaction Lagrangians $\mathcal{L}^{(\mu,\nu)}_+$. μ and ν denote the number of powers of e and v, respectively.

In the following, we first consider a free field and discuss the form of the propagators to be used in perturbation theory. Then we formulate a power counting for Green's functions to find out which vertices of the effective Lagrangian must be used to calculate them to some order in v. Finally, we state how Green's functions can be calculated in a systematic way from the generating functional.

4.5.1 Free Propagators

The propagators to be used in perturbation theory are derived from the Lagrangians $\mathcal{L}^0_{+,\mathrm{HM}}$ and $\mathcal{L}^0_{+,\mathrm{NR}}$. In the notation of appendix A

$$\langle x|D^{-1}_{+,\mathrm{HM}}|y\rangle = \Delta^{\mathrm{HM}}_+(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{E+i\epsilon} \tag{4.72}$$

$$\langle x|D^{-1}_{+,\mathrm{NR}}|y\rangle = \Delta^{\mathrm{NR}}_+(x-y) = -\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{\frac{\mathbf{p}^2}{2M}-E-i\epsilon}, \tag{4.73}$$

where $E = p^0 - M$. The operators in the Lagrangian $\mathcal{L}^{(0,\nu)}_+$ are considered to be corrections to these lowest order propagators. By resumming insertions of $\mathbf{p}^2/2M$ in the HM propagator, we obtain the propagator of the NR scheme.

$$\Delta^{\mathrm{HM}}_+(p)\left(1+\frac{\mathbf{p}^2}{2M}\frac{1}{E}+\left(\frac{\mathbf{p}^2}{2M}\frac{1}{E}\right)^2+\dots\right) = \Delta^{\mathrm{NR}}_+(p). \tag{4.74}$$

Similarly, by including higher and higher corrections and resumming them, we recover the full propagator

$$\Delta_+(p) = -\frac{1}{\omega(\mathbf{p})-p^0-i\epsilon}. \tag{4.75}$$

4.5.2 Naive Power Counting for Green's Functions

We would like to find a way how one can read off the power of v to which a certain Graph contributes. Every Graph can be characterized by the following parameters

E_H	# of external heavy lines
I_H	# of internal heavy lines
I_l	# of internal light lines
$N_{\mu,\nu}$	# of vertices with μ powers of e and ν powers of v
L	# of loops.

In addition, let P denote the power of $1/v$ of the heavy propagator. We have $P=1$ and $P=2$ in the HM and NR schemes, respectively. Excluding external lines, the total power d of v of the diagram is given by

$$d = 4L - PI_H - 2I_l + \sum_{\mu,\nu}\nu N_{\mu,\nu}. \tag{4.76}$$

Using the well known "topological" relations (the factor 2 in front of $N_{\mu,\nu}$ is due

to the fact that at each vertex exactly two heavy lines meet)

$$L = I_H + I_l + 1 - \sum_{\mu,\nu} N_{\mu,\nu} \tag{4.77}$$

$$E_H = \sum_{\mu,\nu} 2N_{\mu,\nu} - 2I_H, \tag{4.78}$$

we get

$$d = 2(L+1) - \frac{2-P}{2} E_H + \sum_{\mu,\nu} N_{\mu,\nu}(\nu - P). \tag{4.79}$$

This formula is certainly correct for $L = 0$ because all factors of v are explicit and there are no integrations over internal momenta. Loops are complicated functions of the external momenta and may produce additional factors of v which can upset this naive power counting. We will briefly come back to this point below and consider only tree graphs for now.

Remember that we always work to a fixed order in the fundamental coupling in e. Therefore, the sum $\sum_{\mu,\nu} \mu N_{\mu,\nu}$ must be the same for every graph contributing to some Green's function. From eq. (4.79) we can see that the leading contribution is given by the graph with as few powers of v as possible. Corrections can be systematically obtained by including vertices with more powers of v.

4.5.3 Perturbation Series

We are now in a position to formulate how a Green's function can be calculated from the generating functional

$$Z[j, j^*, J] = \frac{1}{\mathcal{Z}} \int [dl][dH_+][dH_+^*] e^{iS_+ + \int j^* H_+ + H_+^* j + Jl} \tag{4.80}$$

$$\mathcal{Z} = \int [dl][dH_+][dH_+^*] e^{iS_+} \tag{4.81}$$

$$S_+ = \int d^4x \mathcal{L}_+(x). \tag{4.82}$$

The first step towards the perturbation theory is the separation of the action S_+ into a "free" part (which must be quadratic in the field) and an "interacting" part

$$S_+ = S_+^0 + S_+^{\text{int}}. \tag{4.83}$$

This decomposition depends on the counting scheme and we set

$$S_{+,\text{HM,NR}}^0 = \int d^4x \mathcal{L}_{+,\text{HM,NR}}^0 \tag{4.84}$$

$$S_{+,\text{HM,NR}}^{\text{int}} = S_+ - S_{+,\text{HM,NR}}^0. \tag{4.85}$$

The Gaussian average of some functional F of the fields H_+ and l is denoted by

$$\langle\langle F[H_+, H_+^*, l]\rangle\rangle^{\text{HM,NR}} \doteq \frac{\int [dl][dH_+][dH_+^*] F[H_+, H_+^*, l] e^{iS^0_{+,\text{HM,NR}}}}{\int [dl][dH_+][dH_+^*] e^{iS^0_{+,\text{HM,NR}}}}. \tag{4.86}$$

In particular, the free propagators are given by

$$i\Delta_+^{\text{HM}}(x-y) = \langle\langle H_+(x) H_+^*(y)\rangle\rangle^{\text{HM}} \tag{4.87}$$
$$i\Delta_+^{\text{NR}}(x-y) = \langle\langle H_+(x) H_+^*(y)\rangle\rangle^{\text{NR}} \tag{4.88}$$
$$\frac{1}{i}\Delta_m(x-y) = \langle\langle l(x) l(y)\rangle\rangle. \tag{4.89}$$

The latter is the same in both schemes. In the notation set up in section 3.3.1, the connected n-point functions are written as (we should put indices HM or NR here as well but we suppress them in order to simplify the notation)

$$G_+^{(a,b)}(x,y,z) = \langle\langle \hat{H}_+(x) \hat{H}_+^*(y) \hat{l}(z) e^{iS_+^{\text{int}}}\rangle\rangle_c. \tag{4.90}$$

The perturbation series is obtained by expanding the exponential in powers of v with the constraint that $\sum_{\mu,\nu} \mu N_{\mu,\nu}$ is fixed (see above). After the expansion we are left with Gaussian integrals which can be reduced to sums of products of propagators owing to the Wick theorem.

4.6 Compton Scattering at Tree Level

Let us calculate the tree level truncated Green's function $G_{+,tr}^{(1,2)}$ in the HM scheme to next-to-next-to leading order. Applying formula (4.79), we find the combinations of vertices that yield a specific power of v displayed in table 4.2 We can see,

d	Vertices
-1	$N_{1,0} = 2$
0	$N_{2,0} = 1;\ N_{1,0} = 2, N_{0,2} = 1$
1	$N_{2,1} = 1;\ N_{1,0} = 2, N_{0,2} = 2;\ N_{1,2} = 1, N_{1,0} = 1$

Table 4.2: The combination of vertices that yield a certain power d of v for Compton scattering in the HM scheme.

for example, that the leading term is of $O(1/v)$ (it is just the propagator Δ_+^{HM}) and consists of two vertices of the Lagrangian $\mathcal{L}_+^{(1,0)}$. At $O(v^0)$, we can either use one

vertex from $\mathcal{L}_+^{(2,0)}$ or two from $\mathcal{L}_+^{(1,0)}$ together with $\mathcal{L}_+^{(0,2)}$, which is an insertion of $\mathbf{p}^2/2M$.

The result is

$$\frac{1}{i}G_{+,tr}^{(1,2)} = \frac{-1}{4M^2}\frac{1}{E_q + k_1^0}\left(1 + \frac{(\mathbf{q}+\mathbf{k}_1)^2}{2M(E_q + k_1^0)}\right. \tag{4.91}$$

$$\left. + \frac{(\mathbf{q}+\mathbf{k}_1)^4}{4M^2(E_q + k_1^0)^2} - \frac{\mathbf{q}^2 + \mathbf{p}^2 + 2(\mathbf{q}+\mathbf{k}_1)^2}{4M^2} + O(v^3)\right) \tag{4.92}$$

$$+ \frac{1}{8M^3}\left(1 - \frac{E_q + k_1^0}{2M} + O(v^2)\right). \tag{4.93}$$

According to the truncation rule given in eq. (3.49), we must multiply with

$$\sqrt{2\omega(\mathbf{q})}\sqrt{2\omega(\mathbf{p})} = 2M\left(1 + \frac{\mathbf{q}^2 + \mathbf{p}^2}{4M^2} + O(v^4)\right) \tag{4.94}$$

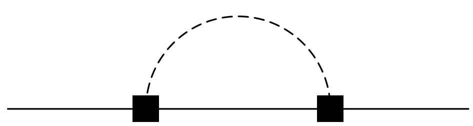
to compare with the truncated Greens function $G_{tr}^{(1,2)}$ of the relativistic theory. Comparing the result with eq. (4.35), we see that

$$G_{tr}^{(1,2)} = \bar{G}_{+,tr}^{(1,2)} \tag{4.95}$$

is true to $O(e^2v^2)$. The amplitude $B_s^{(2)}$ for Compton scattering obtained from $G_{+,tr}^{(1,2)}$ is therefore the same as the one in the relativistic theory with the same precision.

4.7 Power Counting Beyond Tree-Level

Consider the contribution of the graph



to the self energy of the heavy particle, where the boxes represent vertices from $\mathcal{L}_+^{(1,2)}$, i.e. they have two powers of v. According to the formula given in eq. (4.79), this diagram is of $O(v^5)$ in the HM scheme and of $O(v^4)$ in NR. If p is the momentum that flows through the diagram, the loop is a function that depends only on $E = p^0 - M$ and m in the HM scheme but on E, m and $\mathbf{p}$ in the NR scheme. The integrals are of the form

$$I_{\text{HM}} = I_{\text{HM}}\left(\frac{E}{m}\right) \tag{4.96}$$

$$I_{\text{NR}} = I_{\text{NR}}\left(\frac{E}{m}, \frac{\mathbf{p}}{m}, \frac{m}{M}\right). \tag{4.97}$$

The argument of I_{HM} is of $O(1)$ but the one of I_{NR} contains a part that is of $O(v)$. Therefore, the loop destroys the naive power counting in the NR scheme[3]: unless the integral does not really depend on m/M by chance, it produces factors of v which either raise or lower the naive power of v. The former would not be so bad but the latter is a disaster because one must expect that *all* loop graphs start contributing at lowest order.

There is, however, a scenario, where this catastrophe is reduced to a mere inconvenience. If the terms that contribute to a lower order than the naive one are such that they can be absorbed in the coupling constants of the Lagrangian (i.e. polynomials in the energies and momenta), systematic perturbation theory is still possible, because only a finite number of graphs contribute to the "interesting" (non-polynomial) part of the Green's function. The inconvenience is that whenever one pushes the calculation to the next higher order one has to re-match the effective coupling constants (either to the fundamental theory, if possible, or directly to experiment).

It is believed that this is indeed what happens and was checked in an explicit one-loop calculation [27].

In the HM scheme, the problem is absent[4]. However, as mentioned in the introduction, this scheme is not suited for systems where two heavy particles can form a bound state because it leads to spurious infrared divergences, which vanish only upon a resummation of certain contributions (see for example [30]).

[3]This actually depends on the regularization prescription. If one uses a momentum space cutoff $M\alpha \ll \Lambda \ll M$, the loop starts contributing at the naive order see, for example, ref. [7]. In dimensional regularization, however, this is not true

[4]As a side remark, it is interesting to note that this fact is the reason why HBCHPT was introduced to replace the original relativistic treatment of the pion-nucleon system [28], which suffers from the same power counting problem. It was recently shown that a new regularization scheme restores power counting in the manifestly relativistic formulation [29]

Chapter 5

Summary and Outlook

In this work, we have investigated effective theories describing heavy and light scalar particles in the low–energy regime. First, we discussed the concept of the physical mass and of the matching condition for S–matrix elements in a general setting, and then proposed a matching procedure for off–shell Green's functions, that leads – due to different notions of one–particle irreducibility in the original and effective theory – to a specific truncation prescription in the effective theory.

We then investigated these matching conditions for a Yukawa interaction between two heavy (H) and one light field (l). First, we treated the light field as an external source and constructed two non-local Lagrangians that are equivalent to the full theory in the pure particle- and anti-particle sectors. Adding dynamics for the light field, we showed that the amplitudes and properly truncated Green's functions of the effective Lagrangians indeed satisfy the proposed matching conditions to all orders in the coupling in the presence of a UV regulator.

In order to arrive at a local Lagrangian, we discussed the $1/M$ expansion of tree-level scattering. We classified all physical processes by the number of heavy particles and anti-particles in the initial and final states, distinguishing

- *soft* processes: initial and final states contain only heavy particles or only heavy anti particles, like

$$Hl \to Hll,$$

- *semi-hard* processes: both types of particles are present, but their number is separately conserved, like

$$H\bar{H} \to H\bar{H}ll,$$

- *hard* processes: number of particles and anti particles is not conserved separately, e.g.,

$$H\bar{H} \to ll.$$

Starting from the nonlocal Lagrangian, we then constructed the effective local Lagrangian for soft and semi–soft processes at low orders in the $1/M$ expansion.

Hard processes play a special role in this setting: their $1/M$ expansion is difficult, because there is so much energy released that some light particles may become very hard, while others stay soft. Neither did we find a satisfactory treatment of these processes in the literature, nor can we offer one at this moment[1]. Work on the problem is in progress.

Extending the expansion to off-shell Green's functions, we found that there is no natural way to count the energies of heavy particles relative to their momenta (being no longer related through the on-shell condition). We introduced a bookkeeping parameter v and defined two possible counting schemes by assigning powers of it to energies and momenta of the particles and showed how – in a systematic expansion in the fundamental coupling and in the parameter v – tree-level Green's functions can be calculated. We checked the method in the case of Compton scattering.

The final aim of this programme is the application of effective theories to the decay of bound states, like $\pi^+\pi^- \to \pi^0\pi^0$, and to relate these processes to the underlying theory of strong interactions. For this purpose, one needs to include hard processes in the framework, and to set up a consistent and systematic power counting in the scattering sector (including loops) as well as in the bound state calculation where Rayleigh–Schrödinger perturbation theory may be applied. Finally, one has to show how the effective Lagrangians describing QCD at low energies are incorporated in order to arrive at the above described aim. First steps in this direction are already done [3, 4, 7, 19, 20, 31] or will soon be completed [32].

[1] To mention an example, we consider the decay of Ortho– or Parapositronium in the framework of nonrelativistic QED – it requires the inclusion of the hard processes $e^+e^- \to n\gamma$. In the literature, the problem is circumvented by use of a nonhermitean Lagrangian [3]. While this may be useful as far as the calculational purpose is concerned, it is clear that there is room for improving this framework.

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Appendix A

Notation

Metric

We work in Minkowski space with a signature of $(1, -1, -1, -1)$. Three-vectors, denoted by boldface letters, are the three-dimensional parts of contravariant four-vectors

$$x^\mu = \{x^0, x^1, x^2, x^3\} = \{x^0, \mathbf{x}\} \tag{A.1}$$

except for the three-dimensional gradient

$$\nabla = \{\partial_1, \partial_2, \partial_3\}, \tag{A.2}$$

where

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}. \tag{A.3}$$

Fourier Transform

The Fourier transform $f(p)$ of a function $f(x)$ is defined by

$$f(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} f(p). \tag{A.4}$$

Green's functions

Let ϕ be a complex field and x and y denote sets $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ of coordinates. We use the shorthand form

$$\hat{\phi}(x) \doteq \phi(x_1)\phi(x_2)\ldots\phi(x_n). \tag{A.5}$$

The vacuum expectation value of the time ordered product of fields is written as

$$G(x, y) = \langle 0|T\hat{\phi}(x)\hat{\phi}^\dagger(y)|0\rangle. \tag{A.6}$$

Assuming translation invariance, the Fourier transform of G is defined by

$$(2\pi)^4\delta^4(P-Q)G(p,q) = \int d^4x d^4y e^{i\sum_{i=1}^n (p_i x_i - q_i y_i)} G(x,y), \tag{A.7}$$

where $P = \sum_{i=1}^n p_i$ and $Q = \sum_{i=1}^n q_i$. With this convention, the p_i and q_i denote the physical momenta of outgoing and incoming particles if we let the time components x_i^0 and y_i^0 tend to $+\infty$ and $-\infty$, respectively.

In the case of a real scalar field φ, we define

$$G(x) = \langle 0|T\hat{\varphi}(x)|0\rangle \tag{A.8}$$

and ($K = \sum_{i=1}^n k_i$)

$$(2\pi)^4\delta^4(K)G(k) = \int d^4x e^{i\sum_{i=1}^n k_i x_i} G(x). \tag{A.9}$$

Here, the momenta k_i correspond to outgoing particles in the same sense as above.

Operators

Let $\mathbf{O}$ be an operator that acts in some Hilbert space $\mathcal{H}$ of functions defined in Minkowski space. In Dirac notation, the orthogonality and closure relations for the x basis read

$$\langle x|y\rangle = \delta^4(x-y) \tag{A.10}$$

$$\int d^4x |x\rangle\langle x| = \mathbf{1}. \tag{A.11}$$

The x representations of $f \in \mathcal{H}$ and $\mathbf{O}$ are denoted by

$$f(x) = \langle x|f\rangle \tag{A.12}$$

$$O(x,y) = \langle x|\mathbf{O}|y\rangle. \tag{A.13}$$

Accordingly, the action of $\mathbf{O}$ on f reads

$$(\mathbf{O}f)(x) = \int d^4y O(x,y) f(y). \tag{A.14}$$

A differential operator $\mathbf{D}$ has the representation

$$\langle x|\mathbf{D}|y\rangle = \delta^4(x-y) D_y \tag{A.15}$$

so that

$$(\mathbf{D}f)(x) = D_x f(x). \tag{A.16}$$

For any translation invariant operator, i.e. $\langle x|\mathbf{O}|y\rangle = O(x-y)$, we have

$$\Box_x O(x-y) = \Box_y O(x-y). \tag{A.17}$$

If $\mathbf{D}$ is an invariant differential operator (i.e. a function of $\Box$) and $\mathbf{O}$ translation invariant, one may check, using partial integration, that

$$(\mathbf{D}\mathbf{O}f)(x) = (\mathbf{O}\mathbf{D}f)(x). \tag{A.18}$$

Appendix B

Klein-Gordon Green's Functions

A Green's function $G(x)$ of the Klein-Gordon equation is defined by

$$D_M G(x) \doteq (\Box + M^2)G(x) = \delta^4(x) \tag{B.1}$$

together with some boundary conditions. The solution that is a superposition of incoming plane waves for $x_0 < 0$ and of outgoing plane waves for $x_0 > 0$ is the Feynman propagator

$$\begin{aligned}\Delta_M(x) &= \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ipx}}{M^2 - p^2 - i\epsilon} \\ &= i \int \frac{d^3p}{(2\pi)^3 2\omega(\mathbf{p})} e^{i\mathbf{p}\cdot\mathbf{x}} \left(\theta(x^0)e^{-i\omega(\mathbf{p})x^0} + \theta(-x^0)e^{i\omega(\mathbf{p})x^0}\right),\end{aligned} \tag{B.2}$$

where $\omega(\mathbf{p}) = \sqrt{M^2 + \mathbf{p}^2}$.

The Klein-Gordon operator D_M can be decomposed into two first order differential operators

$$\begin{aligned}D_M &= D_+ D_- \\ D_\pm &= \pm i\partial_t - \sqrt{M^2 - \Delta}.\end{aligned} \tag{B.3}$$

The operator

$$d = (2\sqrt{M^2 - \Delta})^{-\frac{1}{2}} \tag{B.4}$$

plays an important role in the construction of the non-relativistic Lagrangian. Its action on a function f is defined through the Fourier representation

$$(df)(x) = \int \frac{d^4p}{(2\pi)^4} \frac{f(p)}{\sqrt{2\omega(\mathbf{p})}} e^{-ipx}. \tag{B.5}$$

The functions

$$\begin{aligned}\Delta_\pm(x) &= -\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ipx}}{\omega(\mathbf{p}) \mp p^0 - i\epsilon} \\ &= -i\theta(\pm x^0) \int \frac{d^3p}{(2\pi)^3} e^{\mp i\omega(\mathbf{p})x^0 + i\mathbf{p}\cdot\mathbf{x}}\end{aligned} \tag{B.6}$$

are Green's functions of $D_\pm$, i.e.

$$D_\pm \Delta_\pm(x) = \delta^4(x) \tag{B.7}$$

and the boundary conditions are chosen such that $\Delta_+(x) = 0$ for $x^0 < 0$ and $\Delta_-(x) = 0$ for $x^0 > 0$. Comparing (B.6) with (B.2) we find

$$\Delta_M(x) = -d^2(\Delta_+(x) + \Delta_-(x)). \tag{B.8}$$

The Green's functions can be viewed as the inverse of the corresponding differential operators. In the notation introduced in appendix A, we write

$$\langle x|D_M^{-1}|y\rangle = \Delta_M(x-y) \tag{B.9}$$

$$\langle x|D_\pm^{-1}|y\rangle = \Delta_\pm(x-y). \tag{B.10}$$

In operator notation, eq. (B.8) can be written in any of the forms (cf. eq. (A.18))

$$\begin{aligned} D_M^{-1} &= -d^2(D_+^{-1} + D_-^{-1}) = -(D_+^{-1} + D_-^{-1})d^2 \\ &= -d(D_+^{-1} + D_-^{-1})d. \end{aligned} \tag{B.11}$$

Finally, with the convention of appendix A, the Fourier transforms are given by

$$\Delta_M(p) = \frac{1}{M^2 - p^2 - i\epsilon} \tag{B.12}$$

$$\Delta_\pm(p) = -\frac{1}{\omega(\mathbf{p}) \mp p^0 - i\epsilon}. \tag{B.13}$$

Appendix C

Canonical Quantization of Free Fields

Let us briefly recall the canonical quantization procedure for a complex scalar field with the Lagrangian

$$\mathcal{L}^0 = \partial_\mu H^* \partial^\mu H - M^2 H^* H. \tag{C.1}$$

The conjugate field is defined by

$$\pi(t, \mathbf{x}) = \frac{\partial \mathcal{L}^0}{\partial \dot{H}(t, \mathbf{x})} = \dot{H}^*(t, \mathbf{x}) \tag{C.2}$$

where $\dot{H}(t, \mathbf{x}) = \partial_0 H(t, \mathbf{x})$. The only non-vanishing Poisson bracket is

$$\{H(t, \mathbf{x}), \pi(t, \mathbf{y})\} = \delta^3(\mathbf{x} - \mathbf{y}) \tag{C.3}$$

and the most general solution of the equation of motion

$$(\Box + M^2) H(x) = 0 \tag{C.4}$$

is a superposition of plane waves

$$H(x) = \int d\mu(p) \left(a(\mathbf{p}) e^{-ipx} + b^*(\mathbf{p}) e^{ipx} \right), \tag{C.5}$$

where the invariant measure is defined by

$$d\mu(p) = \frac{d^3 p}{(2\pi)^3 2p^0} \tag{C.6}$$

and the momentum is on the mass shell

$$p^0 = \omega(\mathbf{p}) = \sqrt{M^2 + \mathbf{p}^2}. \tag{C.7}$$

The factor $(2\pi)^3$ is conventional and is chosen for later convenience. Quantization is performed by replacing H and π by operators which satisfy the equal-time commutation relation

$$[H(t,\mathbf{x}),\pi(t,\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y}). \tag{C.8}$$

The coefficient functions a and b are also operators and obey

$$[a(\mathbf{p}),a^\dagger(\mathbf{q})] = [b(\mathbf{p}),b^\dagger(\mathbf{q})] = 2\omega(\mathbf{p})(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}). \tag{C.9}$$

The operators $a^\dagger$ and $b^\dagger$ can be shown to create one-particle states out of the vacuum

$$|p\rangle = a^\dagger(\mathbf{p})|0\rangle \tag{C.10}$$

$$|\bar{p}\rangle = b^\dagger(\mathbf{p})|0\rangle. \tag{C.11}$$

We shall refer to them as particle and anti-particle states, respectively. The vacuum contains by definition no particles and is defined by the conditions

$$\begin{aligned} a(\mathbf{p})|0\rangle = b(\mathbf{p})|0\rangle = 0 \\ \langle 0|0\rangle = 1. \end{aligned} \tag{C.12}$$

With these conventions, the states are normalized by

$$\langle p|q\rangle = \langle\bar{p}|\bar{q}\rangle = 2\omega(\mathbf{p})(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}). \tag{C.13}$$

Let us apply this formalism to the Lagrangians

$$\mathcal{L}^0_\pm = H^*_\pm(\pm i\partial_t - \sqrt{M^2-\Delta})H_\pm. \tag{C.14}$$

The conjugate fields are

$$\pi_\pm(t,\mathbf{x}) = \frac{\partial\mathcal{L}_\pm}{\partial\dot{H}_\pm} = \pm iH^*_\pm \tag{C.15}$$

and the Poisson brackets are analogous to eq. C.3. The most general solutions of the equations of motion

$$(\pm i\partial_t - \sqrt{M^2-\Delta})H_\pm(x) = 0 \tag{C.16}$$

are

$$H_+(x) = \int d\bar{\mu}(p)a(\mathbf{p})e^{-ipx} \tag{C.17}$$

$$H_-(x) = \int d\bar{\mu}(p)b^*(\mathbf{p})e^{ipx} \tag{C.18}$$

with $p^0 = \omega(\mathbf{p})$. This time, we chose the measure to be

$$d\bar{\mu}(p) = \frac{d^3p}{(2\pi)^3}. \tag{C.19}$$

Replacing the functions by operators, we find

$$[a(\mathbf{p}), a^\dagger(\mathbf{q})] = [b(\mathbf{p}), b^\dagger(\mathbf{q})] = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}). \tag{C.20}$$

They create and destroy particles in the same way as described above. The only difference is the normalization of these states, which now reads

$$\langle p|q\rangle = \langle \bar{p}|\bar{q}\rangle = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}). \tag{C.21}$$

We have sacrificed the rule to label different objects by different symbols to simplify the notation.

Note that the Fock spaces of the theories defined by $\mathcal{L}^0$ and $\mathcal{L}_+ + \mathcal{L}_-$ are identical.

Appendix D

Two-Point Functions

In section 2.1, we consider transition amplitudes in the two models

$$\mathcal{L} = \partial_\mu H^* \partial^\mu H - M^2 H^* H + \bar{\mathcal{L}}^0 + \mathcal{L}^{\text{int}} \tag{2.1}$$

$$\mathcal{L}_+ = H_+^*(i\partial_t - \sqrt{M^2 - \Delta})H_+ + \bar{\mathcal{L}}^0 + \mathcal{L}_+^{\text{int}} \tag{2.16}$$

and show how they can be matched. For this procedure to work, it is necessary that there exists an unambiguous definition of the physical mass of the heavy particle. We must therefore examine the properties of the two-point functions

$$G(p) = \int d^4x e^{ipx} \langle 0|TH(x)H^\dagger(0)|0\rangle \tag{D.1}$$

$$G_+(p) = \int d^4x e^{ipx} \langle 0|TH_+(x)H_+^\dagger(0)|0\rangle. \tag{D.2}$$

It is convenient to express them in terms of one-particle irreducible functions[1] Σ, Σ_+

$$G(p) = \frac{1}{i}\frac{1}{M^2 - p^2 + i\Sigma(p^2) - i\epsilon} \tag{D.3}$$

$$G_+(p) = \frac{1}{i}\frac{1}{\omega(\mathbf{p}) - p^0 + i\Sigma_+(p^0, \mathbf{p}^2) - i\epsilon}, \tag{D.4}$$

with $\omega(\mathbf{p}) = \sqrt{M^2 + \mathbf{p}^2}$. In the absence of interactions, they reduce to the free propagators

$$G(p) = \frac{1}{i}\Delta_M(p) = \frac{1}{i}\frac{1}{M^2 - p^2 - i\epsilon} \tag{D.5}$$

$$G_+(p) = i\Delta_+(p) = \frac{1}{i}\frac{1}{\omega(\mathbf{p}) - p^0 - i\epsilon} \tag{D.6}$$

[1] They can be obtained in perturbation theory from the Legendre transform of the generating functional of connected Green's functions

discussed in appendix B.

The physical mass M_p is defined as the location of the pole of G

$$M_\mathrm{p}^2 = M^2 + i\Sigma(M_\mathrm{p}^2) \tag{D.7}$$

and we can write

$$G(p) = \frac{1}{i}\frac{Z_H}{M_\mathrm{p}^2 - p^2 - i\epsilon} + \text{regular}, p^2 \to M_\mathrm{p}^2, \tag{D.8}$$

where the residue is given by

$$Z_H^{-1} = 1 - i\Sigma'(M_\mathrm{p}^2). \tag{D.9}$$

Let us focus on the pole at $p^0 = \omega_\mathrm{p}(\mathbf{p}) = \sqrt{M_\mathrm{p}^2 + \mathbf{p}^2}$

$$G(p) = \frac{1}{i}\frac{1}{2\omega_\mathrm{p}(\mathbf{p})}\frac{Z_H}{\omega_\mathrm{p}(\mathbf{p}) - p^0 - i\epsilon} + \text{regular}, p^0 \to \omega_\mathrm{p}(\mathbf{p}). \tag{D.10}$$

We may isolate it in a different way by first writing G as

$$G(p) = \frac{1}{i}\Delta_M(p)\left(1 + S(p)\frac{1}{i}\Delta_M(p)\right), \tag{D.11}$$

with

$$S(p) = \frac{\Sigma(p^2)}{1 + \Delta_M(p)i\Sigma(p^2)}. \tag{D.12}$$

The r.h.s. sums up products of propagators with insertions of Σ. This representation shows that the latter is really the 1-particle irreducible two-point function with respect to Δ_M. The idea is to define a new irreducible function *with respect to* Δ_+. It is clear that Σ is still irreducible in this new sense. In appendix B it is shown that

$$\Delta_M(p) = -\frac{1}{2\omega(\mathbf{p})}\left(\Delta_+(p) + \Delta_-(p)\right), \tag{D.13}$$

where $\Delta_-(p) = -1/(\omega(\mathbf{p}) + p^0)$, and we find that S contains new irreducible functions, namely those obtained by connecting factors of Σ with Δ_-, which is considered to be irreducible. Therefore,

$$\begin{aligned}\hat{\Sigma}_+(p^0, \mathbf{p}^2) &\doteq \frac{\Sigma(p^2)}{2\omega(\mathbf{p})} + \frac{\Sigma(p^2)}{2\omega(\mathbf{p})}i\Delta_-(p)\frac{\Sigma(p^2)}{2\omega(\mathbf{p})} + \ldots \\ &= \frac{\Sigma(p^2)}{2\omega(\mathbf{p}) - \Delta_-(p)i\Sigma(p^2)},\end{aligned} \tag{D.14}$$

is the fundamental irreducible function with respect to Δ_+. One may easily check that in terms of $\hat{\Sigma}_+$, S can be written as

$$S(p) = \frac{2\omega(\mathbf{p})\hat{\Sigma}_+(p^0, \mathbf{p}^2)}{1 - \Delta_+(p)i\hat{\Sigma}_+(p^0, \mathbf{p}^2)}. \tag{D.15}$$

Let us also define

$$\hat{G}_+(p) \doteq i\Delta_+(p)\left(1 + \frac{S(p)}{2\omega(\mathbf{p})} i\Delta_+(p)\right) \tag{D.16}$$

$$= \frac{1}{i}\frac{1}{\omega(\mathbf{p}) - p^0 + i\hat{\Sigma}_+(p^0, \mathbf{p}^2) - i\epsilon}. \tag{D.17}$$

The equation

$$\omega_{\mathrm{p}}(\mathbf{p}) = \omega(\mathbf{p}) + i\hat{\Sigma}_+(\omega_{\mathrm{p}}(\mathbf{p}^2), \mathbf{p}), \tag{D.18}$$

which defines the location of the pole of $\hat{G}_+$, is equivalent to (D.7) and we can write

$$\hat{G}_+(p) = \frac{1}{i}\frac{\hat{Z}_+(\mathbf{p}^2)}{\omega_{\mathrm{p}}(\mathbf{p}) - p^0 - i\epsilon} + \text{regular}, p^0 \to \omega_{\mathrm{p}}(\mathbf{p}), \tag{D.19}$$

where

$$\hat{Z}_+^{-1}(\mathbf{p}^2) = 1 - i\hat{\Sigma}'_+(\omega_{\mathrm{p}}(\mathbf{p}), \mathbf{p}^2) \tag{D.20}$$

and the prime refers to the derivative with respect to p^0. With a little algebra, we can cast eq. (D.11) into the form

$$\begin{aligned} G(p) = & \left(1 - \frac{i\hat{\Sigma}_+(p^0, \mathbf{p}^2)}{\omega(\mathbf{p}) + p^0}\right)\frac{\hat{G}_+(p)}{\omega(\mathbf{p}) + p^0} \\ = & \frac{1}{i}\frac{1}{\omega(\mathbf{p}) + p^0}\left(1 - \frac{i\hat{\Sigma}_+(p^0, \mathbf{p}^2)}{\omega(\mathbf{p}) + p^0}\right)\frac{\hat{Z}_+(\mathbf{p}^2)}{\omega_{\mathrm{p}}(\mathbf{p}) - p^0 - i\epsilon} \\ & + \text{regular}, p^0 \to \omega_{\mathrm{p}}(\mathbf{p}), \end{aligned} \tag{D.21}$$

which is to be compared with eq. (D.10). The relation between the residues can be read off to be

$$\hat{Z}_+(\mathbf{p}^2) = \frac{(\omega(\mathbf{p}) + \omega_{\mathrm{p}}(\mathbf{p}))^2}{4\omega_{\mathrm{p}}(\mathbf{p})\omega(\mathbf{p})} Z_H. \tag{D.22}$$

Note that the $\mathbf{p}$ dependence of $\hat{Z}_+$ is entirely due to loop corrections. At tree-level, where $\omega_{\mathrm{p}}(\mathbf{p}) = \omega(\mathbf{p})$, the residues are, of course, both equal to 1.

Let us now come to the original G_+ defined in eq. (D.2). From the previous analysis we find that if we match the irreducible function Σ_+ defined in eq. (D.4) to the full theory according to

$$\Sigma_+(p^0, \mathbf{p}^2) = \hat{\Sigma}_+(p^0, \mathbf{p}^2), \tag{D.23}$$

we also have $G_+(p) = \hat{G}_+(p)$ and the physical mass defined through eq. (D.18) is the same as the one in the relativistic theory. The residues are related by eq. (D.22).

Note that the matching is done off-shell. The only relevant objects for physical quantities are the location of the pole, defining the physical mass in terms of the parameters of the theory, and its residue, providing the effective normalization of the field. Any off-shell matching that does not change these properties is allowed. The construction presented here singles out one of these possibilities rather naturally.

To use these results in a calculation of physical quantities, we must renormalize the theories so that they yield finite results when the regulator is removed. The necessary counter terms at one-loop order are determined in appendix H. The statements derived here can be verified explicitly to this order in perturbation theory.

Appendix E

Reduction Formulae

The reduction formula gives the relationship between the residues of certain Green's functions and physical scattering amplitudes. The underlying assumption is that particles involved in a scattering process behave like free particles long before and long after the collision. This is called the asymptotic condition and must be formulated carefully, see for example [33, 34].

We first give a review of the facts in a relativistic theory and then consider an effective theory which is not manifestly Lorentz invariant.

E.1 Relativistic Theory

We consider the generic Lagrangian

$$\begin{aligned} \mathcal{L} &= \mathcal{L}^0 + \bar{\mathcal{L}}^0 + \mathcal{L}^{\text{int}} \\ \mathcal{L}^0 &= \partial_\mu H^* \partial^\mu H - M^2 H^* H \end{aligned} \tag{E.1}$$

introduced in section 2.1. In the notation of appendix A, connected Green's functions of the heavy field are denoted by

$$G^{(2n)}(x, y) = \langle 0|T\hat{H}(x)\hat{H}^\dagger(y)|0\rangle_c. \tag{E.2}$$

Canonical quantization of the free H field leads to creation and annihilation operators of one particle states as described in appendix C. The asymptotic condition says that the interacting field behaves like a free field in the remote past and future in the weak sense (only for matrix elements)

$$H(x) \overset{x^0 \to -\infty}{\rightarrow} Z_H^{\frac{1}{2}} H_{\text{in}}(x) \tag{E.3}$$

$$H(x) \overset{x^0 \to +\infty}{\rightarrow} Z_H^{\frac{1}{2}} H_{\text{out}}(x) \tag{E.4}$$

The fields H_{in}, H_{out} have all the properties of free fields and Lorentz invariance implies that Z_H is a constant, which is given by the residue of the full two-point

function

$$G^{(2)}(p) = \int d^4x e^{ipx} \langle 0|TH(x)H^\dagger(0)|0\rangle = \frac{1}{i}\frac{1}{M^2 - p^2 + i\Sigma(p^2)}$$
$$= \frac{1}{i}\frac{Z_H}{M_\mathrm{p}^2 - p^2 - i\epsilon} + \dots \tag{E.5}$$

The physical mass M_p and the residue Z_H are defined through the one-particle irreducible function Σ by (see also appendix D)

$$M_\mathrm{p}^2 = M^2 + i\Sigma(M_\mathrm{p}^2) \tag{E.6}$$
$$Z_H^{-1} = 1 - i\Sigma'(M_\mathrm{p}^2). \tag{E.7}$$

We define in- and out states by

$$|p;\text{in}\rangle = a_\text{in}^\dagger(\mathbf{p})|0\rangle$$
$$|p;\text{out}\rangle = a_\text{out}^\dagger(\mathbf{p})|0\rangle \tag{E.8}$$

and similar for $|\bar{p};\text{in}\rangle$, $|\bar{p};\text{out}\rangle$. In fact, the Hilbert spaces of in- and out states are identical and the scattering operator S is the isomorphism that maps a state $|i;\text{in}\rangle$ into the space of out-states

$$|i;\text{in}\rangle = S|i;\text{out}\rangle. \tag{E.9}$$

Defining the T operator by

$$S = 1 + iT, \tag{E.10}$$

the amplitude to find the final state $|f;\text{out}\rangle$ is given by

$$\langle f;\text{out}|i;\text{in}\rangle = \langle f;\text{in}|i;\text{in}\rangle + i\langle f|T|i\rangle$$
$$= \langle f;\text{in}|i;\text{in}\rangle + i(2\pi)^4\delta^4\left(P_f - P_i\right)T_{fi}, \tag{E.11}$$

where we have also defined the T-matrix element T_{fi}. If none of the initial one-particle states are contained in the final state, the first term on the r.h.s. vanishes.

Let's consider a configuration where there are n heavy particles in the initial and final states, i.e.

$$|i;\text{in}\rangle = |q_1,\dots,q_n;\text{in}\rangle \tag{E.12}$$
$$|f;\text{out}\rangle = |p_1,\dots,p_n;\text{out}\rangle. \tag{E.13}$$

Reducing the in- and out states as described, for example, in [33] we find

$$\langle p_1,\dots,p_n;\text{in}|S-1|q_1,\dots,q_n;\text{in}\rangle_c = \left(iZ_H^{-\frac{1}{2}}\right)^{2n}\int d^4x d^4y e^{i\sum_i(p_ix_i - q_iy_i)}$$
$$(\Box_{x_1} + M_\mathrm{p}^2)\dots(\Box_{y_n} + M_\mathrm{p}^2)G^{(2n)}(x,y), \tag{E.14}$$

where the subscript c indicates that disconnected contributions[1] are not included. In terms of the truncated Green's function, this reads

$$\langle p_1, \dots, p_n; \text{in}|S-1|q_1, \dots, q_n; \text{in}\rangle_c = (2\pi)^4 \delta^4 (P-Q)\, Z_H^n\, G_{tr}^{(2n)}(p,q)\Big|_{\text{on-shell}}, \quad \text{(E.15)}$$

where $P = \sum_i p_i, Q = \sum_i q_i$ and "on-shell" means $p_i^0 = \omega_{\text{p}}(\mathbf{p}_i) = \sqrt{M_{\text{p}}^2 + \mathbf{p}_i^2}$, $q_i^0 = \omega_{\text{p}}(\mathbf{q}_i)$. Finally, we read off the expression for the T-matrix element for this process[2]

$$T_{n\to n} = \frac{1}{i} Z_H^n\, G_{tr}^{(2n)}(p,q)\Big|_{\text{on-shell}}. \quad \text{(E.16)}$$

E.2 Effective Theory

Now consider the Lagrangian

$$\begin{aligned} \mathcal{L}_+ &= \mathcal{L}_+^0 + \bar{\mathcal{L}}^0 + \mathcal{L}_+^{\text{int}} \\ \mathcal{L}_\pm^0 &= H_\pm^* D_\pm H_\pm \end{aligned} \quad \text{(E.17)}$$

introduced in sections 2.1.2 and 2.1.3. The Green's functions $G_+^{(2n)}$ are defined in analogy with eq. (E.2). Again, we start with the quantization of the free H_+ field as described in appendix C. It is very important that, up to the normalization, the one-particle state $|p\rangle$ is the same as the one of the relativistic theory: it describes a free scalar particle with momentum $\mathbf{p}$ and energy $\sqrt{M_{\text{p}}^2 + \mathbf{p}^2}$. Therefore, the Fock spaces obtained by applying particle creation operators to the vacuum are the same in both theories. Since the in and out states live in this Fock space, the stage is set for an effective theory that can generate the same transition amplitudes as a relativistic theory (cf. sections 2.1.3 and 2.1.4).

Lorentz symmetry is not respected and the asymptotic condition for the interacting field reads

$$H_+(x) \overset{x^0\to-\infty}{\to} Z_+(\Delta)^{\frac{1}{2}} H_{+,\text{in}}(x) \quad \text{(E.18)}$$

$$H_+(x) \overset{x^0\to+\infty}{\to} Z_+(\Delta)^{\frac{1}{2}} H_{+,\text{out}}(x). \quad \text{(E.19)}$$

The symbol $Z_+(\Delta)$ represents a rotation invariant differential operator. In momentum space, it becomes a function of $\mathbf{p}^2$ and, as in the previous section, we expect it to be the residue of the full two-point function, which, in terms of the irreducible part Σ_+, reads

$$G_+^{(2)}(p) = \frac{1}{i} \frac{1}{\omega(\mathbf{p}) - p^0 + i\Sigma_+(p^0, \mathbf{p}^2) - i\epsilon}. \quad \text{(E.20)}$$

[1] There are no subsets of particles that do not interact

[2] The $T_{n\to n}$ is not precisely the one defined in eq. (E.11) but only the contribution of the connected part

It seems obvious to define the physical mass by the zero of the denominator

$$\omega_{\mathrm{p}}(\mathbf{p}) = \sqrt{M_{\mathrm{p}}^2 + \mathbf{p}^2} = \omega(\mathbf{p}) + i\Sigma_+(\omega_{\mathrm{p}}(\mathbf{p}), \mathbf{p}^2). \tag{E.21}$$

However, for the most general rotation invariant Lagrangian, this would yield a momentum-dependent object M_{p}, which cannot serve as a mass parameter. In view of our goal, which is to reproduce the scattering amplitudes of a fully relativistic theory, we may impose a constraint on the interaction Lagrangian, leading to a momentum-independent parameter M_{p}, which can play the role of the physical mass of the particle. This constraint can be easily implemented in perturbation theory, where it results in a relation among the coupling constants of the theory (see also the discussion in appendix D). Assuming this is done, we can write

$$G_+^{(2)}(p) = \frac{1}{i} \frac{Z_+(\mathbf{p}^2)}{\omega_{\mathrm{p}}(\mathbf{p}) - p^0 - i\epsilon} + \dots \tag{E.22}$$

The residue Z_+ is given by (the prime refers to the derivative with respect to p^0),

$$Z_+^{-1} = 1 - i\Sigma'_+(\omega_{\mathrm{p}}(\mathbf{p}), \mathbf{p}^2). \tag{E.23}$$

In- and out states are defined in analogy to eq. (E.8) and all that was said about the S and T matrix in the previous section applies also here. The fact that H_+ can only destroy a particle in the in-state together with the hermiticity of the Lagrangian implies that the number of heavy particles in the initial and final states must be the same. The procedure of the reduction of in- and out-states can be applied to the effective theory without any problems. The result is

$$\begin{aligned} &\langle p_1, \dots, p_n | S - 1 | q_1, \dots, q_n \rangle_c = \\ &\quad i^{2n} \prod_{i=1}^n Z_+(\mathbf{p}_i^2)^{-\frac{1}{2}} Z_+(\mathbf{q}_i^2)^{-\frac{1}{2}} \int d^4x d^4y e^{i\sum_i (p_i x_i - q_i y_i)} \\ &\quad (\sqrt{M_{\mathrm{p}}^2 - \Delta_{x_1}} - i\partial_{x_1^0}) \dots (\sqrt{M_{\mathrm{p}}^2 - \Delta_{y_n}} + i\partial_{y_n^0}) G_+^{(2n)}(x, y) \end{aligned} \tag{E.24}$$

or, in terms of the truncated function,

$$\begin{aligned} &\langle p_1, \dots, p_n | S - 1 | q_1, \dots, q_n \rangle_c = (2\pi)^4 \delta^4 (P - Q) \\ &\qquad \prod_{i=1}^n Z_+(\mathbf{p}_i^2)^{\frac{1}{2}} Z_+(\mathbf{q}_i^2)^{\frac{1}{2}} \; G_{+,tr}^{(2n)}(p, q)\Big|_{\text{on-shell}} . \end{aligned} \tag{E.25}$$

The notion of on-shell is the same as before and the T matrix element is given by

$$T_{n\to n}^+ = \frac{1}{i} \prod_{i=1}^n Z_+(\mathbf{p}_i^2)^{\frac{1}{2}} Z_+(\mathbf{q}_i^2)^{\frac{1}{2}} \; G_{+,tr}^{(2n)}(p, q)\Big|_{\text{on-shell}} . \tag{E.26}$$

Appendix F

Proof of eq. (3.17)

We prove the decomposition for G_1 in eq. (3.13) – the proof for the decompositions $G_{2,3,4}$ in that equation is very similar. Using

$$A = D_+ + eB \ \ , \ C = D_- + eB \ \ ,$$

one has

$$A - e^2 BC^{-1}B = D_+ + eB - e^2 BC^{-1}B \ \ ,$$

and

$$\begin{aligned} eB - e^2 BC^{-1}B &= eBC^{-1}[C - eB] = eBC^{-1}D_- \\ &= eB(1 + eD_-^{-1}B)^{-1} \ \ , \end{aligned}$$

as a result of which G_1 becomes

$$G_1 = \left[1 + eD_+^{-1}B(1 + eD_-^{-1}B)^{-1}\right]^{-1} D_+^{-1} \ .$$

With

$$1 + eD_+^{-1}B(1 + eD_-^{-1}B)^{-1} = \left[1 + e(D_+^{-1} + D_-^{-1})B\right](1 + eD_-^{-1}B)^{-1} \ \ ,$$

we find

$$G_1 = (1 + eD_-^{-1}B)\left[1 + e(D_+^{-1} + D_-^{-1})B\right]^{-1} D_+^{-1} \ .$$

From

$$G_1 \doteq D_+^{-1} - D_+^{-1}T_{++}D_+^{-1} \ \ ,$$

one has

$$T_{++} = eB\frac{1}{1 + (D_+^{-1} + D_-^{-1})eB} \ .$$

This agrees indeed with

$$dTd = d\frac{1}{1 - elD_M^{-1}}eld \ .$$

Appendix G

Determinants

Consider the generating functional of Green's functions

$$Z[j, j^*, J] = \frac{\int [dH][dH^*][dl] e^{i \int \mathcal{L} + j^* H + H^* j + Jl}}{\int [dH][dH^*][dl] e^{i \int \mathcal{L}}} \tag{G.1}$$

for the toy model

$$\begin{aligned} \mathcal{L} &= \mathcal{L}^0 + \mathcal{L}_l^0 + eH^*Hl \\ \mathcal{L}^0 &= \partial_\mu H^* \partial^\mu H - M^2 H^* H \\ \mathcal{L}_l^0 &= \frac{1}{2}\partial_\mu l \partial^\mu l + \frac{m^2}{2} l^2 \end{aligned} \tag{G.2}$$

studied in chapter 3. Performing the integration over H we get

$$Z[j, j^*, J] = \frac{\int [dl] (\det D_M^{-1} D_e)^{-1} e^{i \int \mathcal{L}_l^0 + j^* D_e^{-1} j + Jl}}{\int [dl] (\det D_M^{-1} D_e)^{-1} e^{i \int \mathcal{L}_l^0}}. \tag{G.3}$$

The operator D_e was introduced in section 3.2

$$D_e \doteq D_M - el \tag{G.4}$$

and $D_M = \Box + M^2$. The factors of $\det D_M$ were added to eq. (G.3) for later convenience.

In this appendix, we first show that

$$\det D_M^{-1} D_e = \det D_+^{-1} \mathcal{D}_+ D_-^{-1} C, \tag{G.5}$$

with the symbols

$$\mathcal{D}_+ = D_+ + eB - e^2 B C^{-1} B \tag{G.6}$$

$$D_\pm = \pm i \partial_t - \sqrt{M^2 - \Delta} \tag{G.7}$$

$$C = D_- + eB \tag{G.8}$$

$$B = dld \tag{G.9}$$

introduced in section 3.2. This decomposition allows us to prove that Z may be written in the form

$$Z[j, j^*, J] = \frac{\int [dl] (\det D_+^{-1} \mathcal{D}_+)^{-1} e^{i \int \mathcal{L}_l^0 + j^* D_e^{-1} j + Jl}}{\int [dl] (\det D_+^{-1} \mathcal{D}_+)^{-1} e^{i \int \mathcal{L}_l^0}}. \tag{G.10}$$

The determinants are ill-defined as long as we don't specify how to deal with the UV divergences inherent in their definitions. In the following we use dimensional regularization and work in $D \neq 4$ dimensions to render all expressions finite. The statements derived here are then valid to all orders in perturbation theory. Actual renormalization to one loop is performed in appendix H.

In the following, we use the propagators Δ_M and $\Delta_\pm$ defined in appendix B as representations of the operators D_M^{-1} and $D_\pm^{-1}$, respectively.

Consider first the l.h.s. of eq. (G.5)

$$\det D_M^{-1} D_e = \det(1 - D_M^{-1} el) = e^{\mathrm{Tr} \ln(1 - D_M^{-1} el)}.$$

Expanding the logarithm we can write

$$\begin{aligned} \mathrm{Tr} \ln(1 - D_M^{-1} el) &= -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}(D_M^{-1} el)^n \\ &= -\sum_{n=1}^{\infty} \frac{1}{n} \int d^D x_1 \ldots d^D x_n \Delta_M(x_1 - x_2) el(x_2) \ldots \Delta_M(x_n - x_1) el(x_1). \end{aligned} \tag{G.11}$$

The n^{th} term of this sum is a loop formed by connecting n fields l by as many propagators Δ_M (see figure G.1). Using the identity $C^{-1} eB = 1 - C^{-1} D_-$,

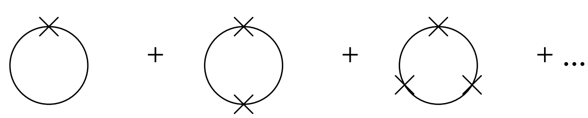


Figure G.1: A graphical representation of the r.h.s. of eq. (G.11). The line stands for a propagator Δ_M and the cross for a light field l.

which follows directly from the definition of C, we can cast the r.h.s. of eq. (G.5) into the form

$$\det D_+^{-1} \mathcal{D}_+ D_-^{-1} C = \det \left(1 + \left(D_+^{-1} + D_-^{-1}\right) eB\right).$$

Proceeding as before, we find

$$\begin{aligned} \mathrm{Tr} \ln \left(1 + \left(D_+^{-1} + D_-^{-1}\right) edld\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int d^D x_1 \ldots d^D x_n \\ d_{x_1} (\Delta_+(x_1 - x_2) + \Delta_-(x_1 - x_2)) d_{x_2} el(x_2) \\ \ldots d_{x_n} (\Delta_+(x_n - x_1) + \Delta_-(x_n - x_1)) d_{x_1} el(x_1), \end{aligned} \tag{G.12}$$

where the subscript of d indicates on which variable it acts. Due to the operator relation

$$D_M^{-1} = -d(D_+^{-1} + D_-^{-1})d \tag{G.13}$$

derived in appendix B this is indeed equal to the l.h.s. of eq. (G.5).

Formally, we can write eq.(G.5) as

$$\det D_M^{-1} D_e = \det D_+^{-1} \mathcal{D}_+ \det D_-^{-1} C. \tag{G.14}$$

Taking the logarithm, we find to first order in e the tadpole term

$$\Delta_M^D(0) \int d^D x l(x) = -\left(d^2 \Delta_+^D(x)\big|_{x=0} + d^2 \Delta_-^D(x)\big|_{x=0} \right) \int d^D x l(x).$$

The explicit expressions of the terms on the r.h.s. are

$$d^2 \Delta_\pm^D\big|_{x=0} = -\int \frac{d^D p}{(2\pi)^D} \frac{1}{2\omega(\mathbf{p})(\omega(\mathbf{p}) \mp p^0 - i\epsilon)}. \tag{G.15}$$

In standard dimensional regularization, where one writes $d^D p = dp^0 d^{D-1}\mathbf{p}$ and integrates over p^0 separately, these are not well defined because the integrand falls off only like $1/p^0$ for large p^0. One may use what is called split dimensional regularization ([35]) where one introduces two independent regulators σ and D according to

$$d^D p = d^\sigma p^0 d^{D-\sigma}\mathbf{p}. \tag{G.16}$$

In this scheme we find

$$d^2 \Delta_+^D(x)\big|_{x=0} + d^2 \Delta_-^D(x)\big|_{x=0} = -e^{i\sigma\frac{\pi}{2}} \frac{M^{D-2}}{(4\pi)^{\frac{D}{2}}} \Gamma(1 - \frac{D}{2}) \tag{G.17}$$

and one can check that this is indeed equal to $-\Delta_M(0)$, evaluated with the same prescription.

This subtlety only occurs in the tadpole. Every other graph has an integrand that falls off at least like $1/(p^0)^2$ and split dimensional regularization coincides with the standard dimensional regularization. Eq. (G.14) is therefore true within this special regularization scheme.

Let us give an intuitive understanding of this decomposition. A loop containing n propagators can be written as a sum of 2^n terms by decomposing Δ_M into $\Delta_\pm$ as in eq. (G.13). One of these terms will exclusively contain anti-particle propagators Δ_- and all of these graphs are collected in the expression $\det D_-^{-1} C$. Therefore, eq. (G.14) can be interpreted as the separation of the contribution of the pure anti-particle sector to loops formed by the heavy field.

To prove eq. (G.10), we show that $\det D_-^{-1}C$ does not contribute to any Green's functions contained in Z. The explicit expression for this determinant is

$$\det D_-^{-1}C = \exp\Big\{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int d^Dx_1\dots d^Dx_n\Delta_-(x_1-x_2)eB(x_2)\dots \\ \times\Delta_-(x_n-x_1)eB(x_1)\Big\}.$$

Upon performance of the l-integration in eq. (G.3) it will produce sub-graphs of the type shown in figure G.2. Because $\Delta_-(p^0,\vec{p}^2)$ contains only one pole in p^0, all

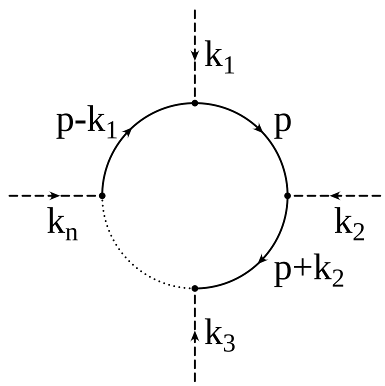


Figure G.2: A typical contribution of a loop formed exclusively with anti-particle propagators (solid lines). It is connected to the rest of the diagram only by propagators of the light field (dashed lines).

poles of the integrand of such a loop lie in the same half-plane. We can close the contour of the integration in the other half-plane and find that the entire integral vanishes, irrespective of the rest of the diagram that this loop is part of. Therefore, we can drop this determinant in the expression for Z *without changing any Green's functions*, which completes the proof of eq. (G.10).

Let us return to the tadpole contribution discussed above. We may separate it by defining

$$\delta \doteq (\det D_M^{-1}D_e)^{-1}e^{-e\Delta_M(0)\int d^Dx l(x)} \tag{G.18}$$

Furthermore, let us add a term to the Lagrangian

$$\bar{\mathcal{L}} = \mathcal{L} - e\frac{1}{i}\Delta_M(0)l. \tag{G.19}$$

The corresponding generating functional

$$\bar{Z}[j,j^*,J] = \frac{\int[dl]\delta e^{i\int \mathcal{L}_l^0 + j^*D_e^{-1}j + Jl}}{\int[dl]\delta e^{i\int\mathcal{L}_l^0}} \tag{G.20}$$

is identical to Z, except that it does not contain any one-loop tadpole contributions. The additional term in the definition of $\bar{\mathcal{L}}$ can be viewed as a 1-loop counter term. We have thus shown that renormalization can be done in such a way that the tadpole is removed from any Green's function (see also appendix H).

Appendix H

1-Loop Renormalization

H.1 Relativistic Theory

We consider the tree-level Lagrangian

$$\mathcal{L} = -H^* D_M H - \frac{1}{2} l D_m l + e H^* H l + j^* H + H^* j + J l, \tag{H.1}$$

where $D_M = \Box + M^2$ and $D_m = \Box + m^2$. It is convenient to replace the complex field H by two real fields ϕ_1, ϕ_2 and the source j by two real sources j_1, j_2 through

$$\begin{aligned} H &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ j &= \frac{1}{\sqrt{2}}(j_1 + i j_2). \end{aligned} \tag{H.2}$$

Renaming $l \equiv \phi_3$, $J \equiv j_3$, we can collect the fields and sources in three-dimensional vectors

$$\begin{aligned} \phi^{\mathrm{T}} &= (\phi_1, \phi_2, \phi_3) \\ j^{\mathrm{T}} &= (j_1, j_2, j_3). \end{aligned} \tag{H.3}$$

The Lagrangian then reads

$$\mathcal{L} = -\frac{1}{2}\phi^{\mathrm{T}} D_0 \phi + \frac{e}{2}(\phi_1^2 + \phi_2^2)\phi_3 + j^{\mathrm{T}}\phi, \tag{H.4}$$

with

$$D_0 = \begin{pmatrix} D_M & 0 & 0 \\ 0 & D_M & 0 \\ 0 & 0 & D_m \end{pmatrix}. \tag{H.5}$$

In units where $\hbar$ is explicit, the generating functional W of connected Green's functions is defined by

$$e^{\frac{i}{\hbar}W[j;\hbar]} = \frac{1}{\mathcal{Z}} \int [d\phi] e^{\frac{1}{\hbar}S[j]}, \tag{H.6}$$

where

$$\begin{aligned} \mathcal{Z} &= \int [d\phi] e^{\frac{i}{\hbar}S[0]} \\ S[j] &= \int d^d x \mathcal{L}(x; j). \end{aligned} \tag{H.7}$$

We use dimensional regularization to give a meaning to the path integral and want to construct the counter term Lagrangian

$$\mathcal{L}_{ct} = \hbar \mathcal{L}^{(1)} + O(\hbar^2) \tag{H.8}$$

that absorbs the divergences in $d = 4$. The expansion of W in powers of $\hbar$ is equivalent to an expansion in the number of loops, so that W_0 and W_1 defined by

$$W[j;\hbar] = W_0[j] + \hbar W_1[j] + O(\hbar^2) \tag{H.9}$$

generate tree- and one-loop graphs, respectively. This expansion is obtained by writing ϕ as fluctuation around the solution $\bar{\phi}$ of the equations of motion

$$\begin{aligned} D_M \bar{\phi}_1 - e\bar{\phi}_1\bar{\phi}_3 - j_1 &= 0 \\ D_M \bar{\phi}_2 - e\bar{\phi}_2\bar{\phi}_3 - j_2 &= 0 \\ D_m \bar{\phi}_3 - \frac{e}{2}(\bar{\phi}_1^2 + \bar{\phi}_2^2)\bar{\phi}_3 - j_3 &= 0. \end{aligned} \tag{H.10}$$

Setting $\phi = \bar{\phi} + \hbar^{1/2}\eta$ and keeping only terms of $O(\hbar)$ we find

$$W_0 = \int d^d x \bar{\mathcal{L}}(x) \tag{H.11}$$

$$W_1 = \frac{i}{2} \ln \frac{\det D}{\det D_0} + \int d^d x \bar{\mathcal{L}}^{(1)}(x), \tag{H.12}$$

where

$$D = D_0 - e \begin{pmatrix} \bar{\phi}_3 & 0 & \bar{\phi}_1 \\ 0 & \bar{\phi}_3 & \bar{\phi}_2 \\ \bar{\phi}_1 & \bar{\phi}_2 & 0 \end{pmatrix} \tag{H.13}$$

and barred quantities are evaluated at $\phi = \bar{\phi}$. Applying the heat kernel technique, the contributions to W_1 that diverge in $d = 4$ can be isolated. The result is

$$\begin{aligned} W_1 = &\frac{e^2}{2}\Delta_1 \int d^d x \left(\bar{\phi}_1^2(x) + \bar{\phi}_2^2(x)\right) + \frac{e^2}{2}\Delta_2 \int d^d x \bar{\phi}_3^2(x) \\ &+ e\Delta_3 \int d^d x \bar{\phi}_3(x) + \text{finite}(d \to 4), \end{aligned} \tag{H.14}$$

with

$$\Delta_1 = \frac{1}{2}\frac{\Gamma(-\omega)}{(4\pi)^{2+\omega}}\left(M^{2\omega} + m^{2\omega}\right) \tag{H.15}$$

$$\Delta_2 = \frac{\Gamma(-\omega)}{(4\pi)^{2+\omega}} M^{2\omega} \tag{H.16}$$

$$\Delta_3 = \frac{\Gamma(-1-\omega)}{(4\pi)^{2+\omega}} M^{2(\omega+1)} \tag{H.17}$$

and $\omega = (d-4)/2$. We introduce the renormalization scale μ with the object

$$\begin{aligned} \hat{L} &= \left(\frac{M}{\mu}\right)^{2\omega} \frac{\mu^{2\omega}}{32\pi^2}\frac{\Gamma(-1-\omega)}{(4\pi)^{\omega}} \\ &= L(\mu) + \frac{\mu^{2\omega}}{32\pi^2}\left(\ln\frac{M^2}{\mu^2} - 1\right) + a(\omega, \frac{M}{\mu}) \end{aligned} \tag{H.18}$$

$$L(\mu) = \frac{\mu^{2\omega}}{32\pi^2}\left(\frac{1}{\omega} - \Gamma'(1) - \ln 4\pi\right). \tag{H.19}$$

The function a vanishes in the limit $\omega \to 0$ and is not needed explicitly. $\hat{L}$ is independent of μ

$$\mu\frac{\partial}{\partial\mu}\hat{L} = 0 \tag{H.20}$$

and so are

$$\Delta_1 = -2\left[L(\mu) + \frac{\mu^{2\omega}}{32\pi^2}\left\{\ln\frac{M^2}{\mu^2} + \frac{1}{2}\ln\frac{m^2}{M^2}\right\} + b(\omega, \frac{M}{\mu}, \frac{m}{\mu})\right] \tag{H.21}$$

$$\Delta_2 = -2\left(L(\mu) + \frac{\mu^{2\omega}}{32\pi^2}\ln\frac{M^2}{\mu^2} + c(\omega, \frac{M}{\mu})\right) \tag{H.22}$$

$$\Delta_3 = 2M^2\hat{L} \tag{H.23}$$

Like a, the functions b and c vanish for $\omega \to 0$. In order to cancel these divergences, we need a counter term Lagrangian of the form

$$\mathcal{L}^{(1)} = -\frac{e^2}{2}c_1\left(\phi_1^2 + \phi_2^2\right) - \frac{e^2}{2}c_2\phi_3^2 - c_3 e M^2 \phi_3, \tag{H.24}$$

The dimensionless constants c_n can be chosen to be independent of μ and in the $\overline{MS}$ scheme we set

$$c_n = c_n^r(\mu, \omega) + \Gamma_n L(\mu). \tag{H.25}$$

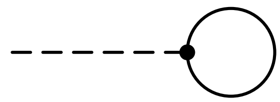
The renormalized couplings c_n^r are finite and depend on the scale according to the renormalization group equations

$$\mu\frac{\partial}{\partial\mu}c_n^r(\mu, \omega) = -2\omega\Gamma_n L(\mu). \tag{H.26}$$

From eq. (H.14) we can read off

$$\Gamma_{1,2} = -2. \tag{H.27}$$

The term Δ_3 plays a special role. In appendix G it was identified with the loop of the tadpole graph



which is simply the Fourier transform $\Delta_M(0)$ of the heavy propagator at zero momentum. In fact, we have $\Delta_3 = -i\Delta_M(0)$. Now, in that appendix it was shown that by adding the term $ie\Delta_M(0)l$ to the Lagrangian, the tadpole is removed from all the Green's functions (see eqns. (G.18) through (G.20)). We therefore chose

$$c_3 = 2\hat{L}. \tag{H.28}$$

Physical quantities can be expressed in terms of the scale-independent and finite couplings

$$\bar{c}_n = -c_n^r(\mu, 0) + \frac{\Gamma_n}{32\pi^2} \ln \frac{M^2}{\mu^2}. \tag{H.29}$$

They are determined through the condition that the parameters M and m should coincide with the physical masses M_p and m_p. The explicit expressions are not needed here.

Finally, we may go back to the original fields and find that

$$\mathcal{L}(H, H^*, l) - c_1 e^2 H^* H - c_2 \frac{e^2}{2} l^2 - c_3 e M^2 l \tag{H.30}$$

gives finite results in $d = 4$ at 1-loop level.

H.2 Effective Non-local Theory

In appendix G it is shown that the non-local Lagrangian

$$\mathcal{L} = \mathcal{L}_+ + j^* H_+ + H_+^* j + Jl \tag{H.31}$$

constructed in section 3.3 contains the same loops as the full theory. Therefore, the only divergent graphs to one loop are the self-energies and the vacuum polarization of the light field.

Consider first the two-point function of the light field. We know that it is identical in both theories (this is evident by comparing the expressions (3.26) and (3.29) of the generating functionals) and so must be the counter terms.

By comparing the two-point functions of the heavy field in eqns. (3.42) and (3.43) we find that $-e^2 c_1 (dH_+^*)(dH_+)$ is the counter term needed to render the self-energy of H in the effective theory finite.

The vacuum expectation value of the light field is given by

$$v = \int d^4x e^{ipx} \langle 0|l(x)|0\rangle = (2\pi)^4 \delta^4(p)\Delta_m(p) ie[d^2\Delta_+](0) \tag{H.32}$$

where as in the relativistic theory we have

$$v = (2\pi)^4 \delta^4(p)\Delta_m(p) ie\Delta_M(0). \tag{H.33}$$

The quantity $[d^2\Delta_+](0)$ is evaluated in split dimensional regularization discussed in appendix G

$$\begin{aligned} [d^2\Delta_+](0) &= -\frac{1}{2} e^{i\sigma\frac{\pi}{2}} \Delta_3 \\ &= -M^2 e^{i\sigma\frac{\pi}{2}} \hat{L}. \end{aligned} \tag{H.34}$$

The appropriate counter term is therefore $-\tilde{c}_3 e M^2 l$ with

$$\tilde{c}_3 \doteq e^{i\sigma\frac{\pi}{2}} \hat{L}. \tag{H.35}$$

Putting everything together, the effective Lagrangian that is finite at 1-loop is given by

$$\mathcal{L}_+ - c_1 e^2 (dH_+^*)(dH_+) - c_2 \frac{e^2}{2} l^2 - \tilde{c}_3 e M^2 l. \tag{H.36}$$

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